ASYMPTOTICALLY GAUSSIAN DISTRIBUTION FOR RANDOM PERTURBATIONS OF ROTATIONS OF THE CIRCLE

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Let $T_{\varepsilon, \omega}$ be a transformation of the two-dimensional torus $\mathbb{T}^2$ given by the formula $T_{\varepsilon, \omega} : (x, y) \rightarrow (2x, y + \omega + \varepsilon x) \mod 1$. A version of the functional central limit theorem is formulated for variables of the form $n^{-1/2} \sum_{k=0}^{n-1} f \circ T_{\varepsilon, \omega}^k$, where $\varepsilon$ is an irrational number and $f$ belongs to a class of real-valued functions on $\mathbb{T}^2$ described in terms of $\varepsilon$. The proof will be published elsewhere. Bibliography: 7 titles.

S. Siboni [7] introduced the family $\{T_{\varepsilon, \omega}\}$ of transformations of the two-dimensional torus $\mathbb{T}^2$, where $\varepsilon \in \mathbb{R}$ and $\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, defined by

$$T_{\varepsilon, \omega} : (x, y) \rightarrow (2x, y + \omega + \varepsilon x) \mod 1.$$ 

Here $x, y$ are coordinates on the torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ mapping it onto $[0, 1) \times [0, 1)$. Thus, $T_{\varepsilon, \omega}$ is a skew product with the endomorphism $S : x \mapsto 2x$ of the circle $\mathbb{T}$ in the base and with a rotation by an angle (depending on $x$) in a fiber (also a circle) over a point $x \in \mathbb{T}$. The Haar measure on $\mathbb{T}^2$ is invariant with respect to $T_{\varepsilon, \omega}$. This family was proposed in [7] as a model of a random perturbation of frequency in a Hamiltonian single-frequency system. It was also proved there that $T_{\varepsilon, \omega}$ is a mixing transformation if and only if $\varepsilon$ is irrational. As to further investigation of ergodic and probabilistic properties, in particular, of the central limit theorem, these questions are mentioned in [7] (for example, a physical interpretation of the variance of the limiting Gaussian distribution in a hypothetical central limit theorem is given), but left open. W. Parry proved in [5] that $T_{\varepsilon, \omega}$ is an exact endomorphism if $\varepsilon$ is irrational. A natural question in the framework of measure-theoretical classification of the transformations $T_{\varepsilon, \omega}$ is whether any such transformation with irrational $\varepsilon$ is a unilateral Bernoulli shift. It turned out that this question is not trivial. Under some additional assumptions imposed on $\varepsilon$, the affirmative answer was given in [6].

In the present note, a version of the functional limit theorem for $T_{\varepsilon, \omega}$ is announced. The proof is based on the martingale approximation method which goes back to [2]. This approach allows us to bypass the basic problems of the ergodic theory of transformations $T_{\varepsilon, \omega}$ and only requires irrationality of $\varepsilon$. The class of functions to which our theorem can be applied depends on the Diophantine properties of $\varepsilon$.

MAIN STATEMENT

Let $(X, \mathcal{M}, P)$ be a probability space; let $T : X \rightarrow X$ be a measurable $P$-preserving transformation, and $f$ a measurable real-valued function. We construct, for any $n \geq 1$, a random piecewise-linear function $J_n f$ on $[0, 1]$ that is linear on intervals of the form $[(m-1)/n, m/n]$ and takes value $n^{-1/2} \sum_{k=0}^{n-1} f \circ T^k$ at $m/n$ for $m = 1, \ldots, n$ (we mean that $J_n f(0) = 0$). We say that $f$ satisfies the functional central limit theorem with limiting variance $\sigma^2$ if the distribution of the random piecewise-linear functions $J_n f$, regarded as a measure on the space $C[0, 1]$ of continuous functions on $[0, 1]$, weakly converges to the Wiener measure with parameter value $\sigma^2$ as $n \rightarrow \infty$. Here the parameter of the Wiener measure is the variance of the value $w(1)$ of the corresponding Wiener process $w(\cdot)$, and weak convergence means here, as usual, the convergence of integrals for any bounded continuous function.

Now set $X = \mathbb{T}^2$, $\mathcal{M} = \mathbb{B}^2$, the Borel $\sigma$-field of $\mathbb{T}^2$, $T = T_{\varepsilon, \omega}$ for certain $\omega$ and $\varepsilon$, and let $P$ be the Haar measure on $\mathbb{T}^2$ (an integral with respect to it will be written in the form $\int f(x, y) dx dy$). $L_2$-spaces of functions with respect to the Haar measure on $\mathbb{T}^2$ and $T$ are denoted, respectively, by $L_2(\mathbb{T}^2)$ and $L_2(T)$.

We define $f_k \in L_2(T)$, for any $f \in L_2(\mathbb{T}^2)$ and $k \in \mathbb{Z}$, by the relation

$$(f_k)(x) = \int_{T} \exp(-2\pi i k y) f(x, y) dy,$$
where the integral is calculated with respect to the Haar measure on $\mathbb{T}$. For any $k \in \mathbb{Z}$, an operator $W_k = W_{k, \epsilon, \omega}$ acts on $L^2(\mathbb{T})$ according to

$$(W_k g)(x) = \frac{1}{2} \left( g\left(\frac{x}{2}\right) \chi_k \left(-\frac{x}{2} - \epsilon\right) + g\left(\frac{x+1}{2}\right) \chi_k \left(-\frac{x+1}{2} - \epsilon\right) \right),$$

where $\chi_k(x) = \exp(2\pi i k x)$.

Further, let $\mathcal{B}_n$ be a $\sigma$-field of subsets of $\mathbb{T}$ generated by all intervals of the form $\{x : l/2^n \leq x < (l + 1)/2^n\}$, where $0 \leq l \leq 2^n - 1$, and let $L_2(\mathcal{B}_n)$ be a subspace of $L_2(\mathbb{T})$ consisting of all $\mathcal{B}_n$-measurable functions.

We denote by $E(\cdot | \mathcal{B}_n)$ the conditional expectation operator with respect to $\mathcal{B}_n$.

**Theorem.** If a real-valued function $f \in L^2(\mathbb{T}^2)$ and an irrational real number $\epsilon \in \mathbb{R}$ are such that the following assumptions hold:

1. \( \int_{\mathbb{T}^2} f(x, y) \, dx \, dy = 0 \),
2. \( \sum_{k \in \mathbb{Z}} \left( \sum_{n \geq 0} ||f_k - E(f_k | \mathcal{B}_n)||_{L^2(\mathbb{T})} \right)^2 < \infty \),
3. \( \sum_{k \in \mathbb{Z}} k^2 ||f_k||^2_{L^2(\mathbb{T})} < \infty \),
4. \( \sum_{k \in \mathbb{Z}, k \neq 0} < 2k\epsilon >^{-4} ||f_k||^2_{L^2(\mathbb{T})} < \infty \)

(\(< \cdot >\) denotes the distance to the nearest integer), then for the function $f$ and transformations $T_{k, \epsilon}$ the functional central limit theorem is valid. The parameter $\sigma^2$ of the limiting Gaussian process is given by

$$
\sigma^2 = \sum_{k \in \mathbb{Z}} \left( \sum_{n \geq 0} ||W_k f_k||^2_{L^2(\mathbb{T})} - \sum_{n \geq 1} ||W_k f_k||^2_{L^2(\mathbb{T})} \right),
$$

where the series on the right-hand side is absolutely convergent.

**Corollary.** The functional central limit theorem may be applied if, besides assumptions (1) and (2) of the theorem, the following requirements (5) and (6) are also satisfied:

5. \( \epsilon > 0, \sum_{k \in \mathbb{Z}} k^4 + 4 ||f_k||^2_{L^2(\mathbb{T})} < \infty \);
6. \( \epsilon \in \mathbb{R} \) is such that for some $\delta > 0$ and $C = C(\epsilon, \delta) > 0$ the inequality $|qe - p| > C |q|^{-(1+\delta)}$ holds for any $p$ and $q \in \mathbb{Z}, q > 0$.

It is obvious that (3) follows from (5), and (4) follows from (5) and (6).

We note that (5), in turn, is satisfied for any irrational algebraic $\epsilon$ (a theorem by Thue--Siegel--Roth, see [4]), and also for almost all $\epsilon \in \mathbb{R}$ (as a consequence of a stronger measure-theoretical result in Diophantine approximation theory [4]). Under the assumptions of the theorem stated above, a more detailed conclusion can also be made which shows that the separate harmonics in the expansion $f(x, y) = \sum_{k \in \mathbb{Z}} f_k(x) \exp(2\pi i k y)$ give independent contributions to the limiting Gaussian distribution. The precise statement of this assertion and the proof will be published elsewhere.

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**REFERENCES**