TWO INEQUALITIES FOR PARAMETERS OF A CELLULAR ALGEBRA

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Two inequalities are proved. The first is a generalization for cellular algebras of a well-known theorem about the coincidence of the degree and the multiplicity of an irreducible representation of a finite group in its regular representation. The second inequality that is proved for primitive cellular algebras gives an upper bound for the minimal subdegree of a primitive permutation group in terms of the degrees of its irreducible representations in the permutation representation. Bibliography: 11 titles.

1. INTRODUCTION

In the representation theory of finite groups, permutation representations are of special importance. This can be partly explained by the fact that permutation groups were historically the first form of general group theory. It is more important, however, that the space of a permutation representation have a natural linear base. The nature of this base enables us to use a combinatorial-algebraic language for the description of irreducible representations entering the permutation representation. Such an approach is not restrictive, since the regular representation of a group is a permutation representation and each irreducible representation enters the regular one.

Let $G$ be a permutation group on a finite set $V$. The permutation representation $\rho : G \to \text{End}(\mathbb{C}^V)$ associates to each permutation $g \in G$ the permutation matrix $\rho(g) \in \text{Mat}_V$. The centralizer algebra $Z(G, V)$ consisting of all matrices from $\text{Mat}_V$ that commute with any matrix $\rho(g)$, $g \in G$, has a natural representation $Z(G, V) \to \text{End}(\mathbb{C}^V)$. The irreducible representations of $G$ entering $\rho$ are in 1-to-1 correspondence with the irreducible representations of the algebra $Z(G, V)$, and the degree of any representation of $G$ coincides with the multiplicity of the corresponding representation of $Z(G, V)$, and vice versa. Thus, from the algebraic point of view, the study of the irreducible representations entering the permutation representation of a group is reduced to the study of the irreducible representations of the corresponding centralizer algebra.

One of the first applications of the centralizer algebras to permutation group theory was related to the characterization of $B$-groups (i.e., the groups $H$ with the following property: if a primitive group contains a regular subgroup isomorphic to $H$, then it is 2-transitive (see [11])). In [9], I. Schur proved that any cyclic group of composite order is a $B$-group. Schur's method consisted of studying the subalgebras $W$ of the centralizer algebra $Z(H, V)$, with $H$ acting regularly on $V$, that have the following three properties:

\begin{align*}
(C1) \ I_V, J_V \in W; \\
(C2) \ \text{if } A \in W, \text{ then } A^* \in W; \\
(C3) \ \text{if } A, B \in W, \text{ then } A \circ B \in W,
\end{align*}

where $I_V$ is the identity matrix, $J_V$ is the matrix with all elements equal to one, $A^*$ is the Hermitian conjugate to $A$, and $A \circ B$ is the Hadamard (componentwise) product of the matrices $A$ and $B$.

In 1968, B. Weisfeiler and A. Lehman, while investigating the Graph Isomorphism Problem, found it useful to consider matrix algebras $W$ satisfying conditions (C1), (C2), and (C3) (see [10]). They called such algebras cellular since condition (C3) implies the existence in $W$ of a linear base $\mathcal{R}$ consisting of $\{0,1\}$-matrices and of a partition $V = V_1 \cup \ldots \cup V_s$ (into cells $V_1, \ldots, V_s$) such that all nonzero elements of any matrix belonging to $\mathcal{R}$ are concentrated at the intersections of the rows and the columns whose indices belong to some $V_i$ and $V_j$, respectively. Thus, to each cell $V_i$ there corresponds a cellular subalgebra of the algebra $W$ consisting of the matrices with support $V_i$. This subalgebra is called a homogeneous component of $W$. For example, the cells of the algebra $W = Z(G, V)$ coincide with the orbits of the group $G$, and its homogeneous components are the centralizer algebras of the transitive constituents of $G$.


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It should be noted that, from the algebraic point of view, a cellular algebra is just a specific representation of a finite-dimensional semisimple algebra. From the combinatorial point of view, the existence of a special base enables us to regard this algebra as a colored graph so that the adjacency matrix of the graph with edges of the same color belongs to \( \mathcal{R} \). Such colored graphs were independently discovered by D. Higman in [7] in connection with the existence problem for combinatorial objects and were called coherent configurations. Two special classes of coherent configurations are formed by association schemes and block designs [1].

In this paper, we prove two inequalities connecting the main algebraic and combinatorial invariants of a cellular algebra. The first theorem generalizes a well-known fact that the multiplicity of any irreducible representation of a finite group in its regular representation coincides with the degree of this representation (this is the case of a regular cellular algebra, i.e., the centralizer algebra of a regular permutation group). For an arbitrary cellular algebra \( W \subset \text{Mat}_V \), the role of the regular representation is played by the standard representation \( \Delta_{\text{stand}} \) of the algebra \( W \) defined by the inclusion of \( W \) into \( \text{Mat}_V \). By (C2), the algebra \( W \) is semisimple and its standard representation is completely reducible, i.e.,

\[
\Delta_{\text{stand}} = \sum_{\pi} m_{\pi} \cdot \pi,
\]

where \( \pi \) is an irreducible representation of the algebra \( W \) and \( m_{\pi} \) is its multiplicity in \( \Delta_{\text{stand}} \).

**Theorem 1.** For any irreducible representation \( \pi \) of a cellular algebra \( W \) the following inequality holds:

\[
n_{\pi} \leq s(\pi) m_{\pi},
\]

where \( n_{\pi} = \deg(\pi) \) is the degree of the representation \( \pi \) and \( s(\pi) \) is the number of the homogeneous components of the algebra \( W \) on which \( \pi \) is not identically zero. Moreover, the equality is simultaneously attained for all irreducible representations if and only if the cellular algebra \( W \) is quasiregular (i.e., any of its homogeneous components is regular).

**Corollary.* Let \( W \) be a homogeneous cellular algebra. Then

\[
n_{\pi} \leq m_{\pi}
\]

for all irreducible representations \( \pi \) of \( W \). The equality is simultaneously attained for all irreducible representations if and only if the algebra \( W \) is regular.

It is well known that each finite group can be constructed by extensions from simple groups. In this sense the role of primitive groups for permutation group theory is similar to the role of simple groups in the theory of finite groups. However, although all simple groups are known, the classification of primitive groups is far from being completed. The case of primitive cellular algebras seems to be considerably more difficult, and their structure is almost unstudied (the exact definition of a primitive cellular algebra can be found in Sec. 2; here we only note that a special class of them is formed by the centralizer algebras of primitive groups).

The second and third theorems of this paper generalize an upper bound for the minimal subdegree \( d \) of a primitive permutation group \( G \) to cellular algebras. Here we mean the estimation \( d \leq J(m) \), where \( m \) is the minimal degree of a nonprincipal irreducible representation of \( G \) entering its permutation representation, and \( J \) is the Jordan function. (Sketch of the proof: it follows from the primitivity of \( G \) and Jordan's theorem that \( G \) has a normal Abelian subgroup \( A \) of index not exceeding \( J(m) \); therefore, \( d \leq |G|/|V| \leq [G:A] \leq J(m) \).)

To formulate the statements, we define a basis relation of a cellular algebra \( W \subset \text{Mat}_V \) as a binary relation on \( V \) whose adjacency matrix belongs to \( \mathcal{R}(W) \). In the homogeneous case, each basis relation \( R \) induces a directed graph for which the in-degree and the out-degree of any vertex are equal to the same number \( d(R) \), called the degree of \( R \).

*When the paper had already been written, the authors noticed that the inequality presented below also follows from Corollary 8 of [4] and paper [2] in this volume.