THE ASYMPTOTIC VARIANCE OF A TIME AVERAGE IN A BIRTH–DEATH PROCESS

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Abstract

The variance of a time average is important for planning, running and interpreting experiments. This paper derives a simple method to find this variance for the case of a Markov process. This method is then applied to obtain the variance of a time average for the case of a birth-death process.

Keywords and phrases

Time-average, variance, estimation, birth-death process.

1. Introduction

This paper deals with the asymptotic distribution of a time average in an equilibrium birth-death process. Specifically, let $X(t)$, $t \geq 0$ be a birth-death process in equilibrium, that is

$P \{ X(0) = i \} = P \{ X(t) = i \} = \pi_i, \quad t \geq 0.$

Then, one can define a functional $H(t) = f_H(X(t))$, and calculate the time average of $H(t)$ from 0 to $T$ as

$\overline{H}(T) = \frac{1}{T} \int_0^T H(t) \, dt = \frac{1}{T} \int_0^T f_H(X(t)) \, dt.$

$H(t)$ could for instance be a cost associated with $X(t)$, and $\overline{H}(T)$ would then represent the average cost per time unit. Alternatively, one could define $H(t)$ to be 1 if
$X(t) = 0$, and zero otherwise. In this case, $\bar{H}(T)$ would represent the proportion of time the system is in state $0$. Of course, $H(t)$ can also be equal to $X(t)$, and in this case, $\bar{H}(T)$ is the time average of $X(t)$. Other usages of $H(t)$ are also possible.

Keilson and Rao [11] show that in irreducible Markov processes with finite state space, $\bar{H}(T)$ is asymptotically normal, provided the second moment of $H(t)$ is finite. The asymptotic distribution of $\bar{H}(T)$ is thus determined by its expectation and its variance. The expectation of $\bar{H}(T)$, we call it $L_H$, must in this case be equal to that of $H(t)$ in equilibrium. If the states of the process are numbered from 0 to $N$, one has thus

$$E(\bar{H}(T)) = E(H(t)) = \sum_{i=0}^{N} f_H(i) \pi_i = L_H.$$ 

There is an extensive literature on the variance of time averages and related areas. For references on the different aspects of this topic, the reader may consult the review paper by Reynolds [15]. Further references are given by Halfin [9], Woodside et al. [19], and Glynn [6]. In our connection, we are mainly interested in research papers dealing with variances of time averages in Markov chains, birth-death processes, and Markovian queues. Many investigators, in particular Reynolds [17], Halfin [9] and, though indirectly, also Gaffarian and Ancker [4], have used spectral analysis to find the variances in question. It is our opinion, and that of others (Woodside et al. [19]) that this approach will not normally be convenient for numerical calculations. Kemeny and Snell and, following them, Keilson and Rao, have solved the problem, using the fundamental matrix. The calculation of this matrix requires one to calculate an inverse of a matrix of size $N$ by $N$, where $N$ is the number of states. The calculation of an inverse requires in general $2N^3$ operations, and it is not immediately obvious how the number of operations can be reduced for the special case of the birth-death process. If there are many states, this approach thus becomes costly, even in the case of birth-death processes. In this paper, we present a method which uses the special structure of the birth-death process to reduce the number of calculations required to obtain the variance to $7N$, where $N$ is again the number of states. We investigate processes with up to 201 states (see table 1), and in this case, the number of operations can be reduced by a factor of over 10 000, which is substantial.

Gebhard [5] and, independently, Jenkins [10] found a very simple formula for the variance of the average number in the system in an $M/M/1$ queue. Different, but more complex formulae for this variance were later derived by Reynolds [16] and Woodside et al. [19]. In this paper, we not only derive a formula for the $M/M/1$ queue, but also for the $M/M/c$ queue, which is substantially more difficult. Finally, Burman [2] presented a formula for calculating the variance in question for the case of a birth-death process.