INSTABILITIES IN BUOYANCY-DRIVEN BOUNDARY-LAYER FLOWS IN A STABLY STRATIFIED MEDIUM

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Abstract. An analysis is made of the stability of a buoyancy boundary layer existing on an inclined wall which is either heated or cooled relative to ambient, stably stratified fluid. A Boussinesq fluid, with various Prandtl numbers, is considered. Detailed calculations of the linear stability boundaries are made for both streamwise periodic, travelling disturbances and spanwise periodic, stationary disturbances. The former type is found to become unstable first for all angles of tilt, but calculations at a particular angle indicate that the latter can have a higher growth rate once the Reynolds number is sufficiently above the critical value. Energy integrals are evaluated at the critical Reynolds number for various angles of tilt in order to clarify the mechanisms for energy transfer to the disturbance.

1. Introduction

Flows driven by buoyancy forces occur very commonly in nature and in a number of technological processes. An important example is the flow over heated (or cooled) mountain slopes, which is described by Prandtl (1952) and which is characterized by a boundary layer of constant thickness, the boundary-layer growth being inhibited by the presence of an external stably stratified fluid. Such layers representing a balance between buoyancy and viscous forces have been called 'buoyancy layers' and are analogous to the 'Ekman layers' in rotating systems which represent a balance between Coriolis and viscous forces. Flows induced along sloping surfaces near shores of oceans due to a buoyancy stratification caused by a salinity gradient constitutes another system where buoyancy layers occur (Phillips, 1970; Wunsch, 1970).

The present paper will be concerned with the stability of such flows with respect to infinitesimal disturbances. The problem is distinct from the usual stratified shear flow stability problem due to the existence of density gradients both normal to and in the direction of flow. Linear theory is used, and the stability is investigated with respect to both transverse (streamwise periodic) travelling waves and longitudinal (spanwise periodic) stationary disturbances. Gill and Davey (1969) considered the stability of the buoyancy layer developed over a heated vertical plate in the presence of an external stably stratified system. Two-dimensional travelling disturbances were only considered even though a Squire-type theorem cannot be shown analytically to hold good for this system. The presence of the 'nose' in the neutral curve for Prandtl number 0.72, characteristic of buoyancy-driven stabilities, was discussed. The mechanism of instability was shown to be mechanically driven for low Prandtl numbers, the disturbance deriving its kinetic energy mainly from the mean shear,
while for high Prandtl numbers the instability was buoyancy driven, the disturbance deriving its kinetic energy mainly from the buoyancy forces. The analysis of a buoyancy layer for a vertical plate includes the effect of the streamwise external density gradient. When the plate is inclined, however, the component of the external stratification parallel to the streamwise direction is diminished, while a component of external stratification normal to the wall is simultaneously established. One would expect this normal component of external stratification to be stabilizing. The streamwise component of external stratification can have unique effects. For forced convection flows between two horizontal parallel plates, in the presence of a streamwise gradient along the plates, Nakayama et al. (1971) showed the destabilizing effect of bottom heating on the critical Rayleigh number for longitudinal rolls. For the case of top heating (stably stratified), the destabilizing effect of a sufficiently large streamwise gradient along the plates was found to generate instability also in the form of longitudinal rolls (even for the case of zero Prandtl number, when no distortion of the basic temperature field occurs). This novel instability has also been discussed by Faller (1971). Hart (1971) studied the stability of free convection flows in an inclined slot for various situations. Both travelling transverse waves and spanwise periodic disturbances were considered. The destabilizing effect of the streamwise gradient in the core region was shown to be capable of causing instability in the form of longitudinal rolls for the case of top heating, and the theoretical results were largely confirmed by experiment. Hart, however, ignored the existence of a temperature gradient normal to the wall in the core region. The present analysis includes the effect of the normal component of external stratification, which is a given parameter here in contrast to the stratification in the core for the slot problem, where it is a function of the imposed temperature difference and the aspect ratio of the slot. In the present problem, of course, no upper wall exists, and a unidirectional boundary-layer flow develops for any non-zero $\Delta T$.

2. Analysis

The system under consideration is shown in Figure 1. A stably stratified fluid with kinematic viscosity $v_0$, coefficient of thermal expansion $\alpha_0$, and thermal diffusivity $\kappa_0$, fills the space far away from an inclined wall which is heated (or cooled) with respect to the surroundings. The wall temperature is raised (or lowered) by a fixed amount $\Delta T=|A-T_0|$ above (or below) that of the fluid outside the boundary layer. This requires a streamwise variation of the wall temperature to correspond with a similar variation in the surrounding fluid, as shown in Figure 1. The inclination angle of the plate with respect to the vertical is $\theta$. Coordinates $\hat{s}$ and $\hat{n}$ are parallel and normal to the wall, respectively. The width of the buoyancy layer developed on the wall is of order $\delta$ and streamwise velocities of order $\hat{u}$ are generated where, as mentioned by Prandtl (1952),

$$\delta = \sqrt[4]{\frac{4v_0\kappa_0}{\alpha_0\beta_0 T_0 \cos^2 \theta}}$$  (2.1)