On the spectrum of resolvable orthogonal arrays invariant under the Klein group $K_4$

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Abstract. It is shown that there exists a resolvable $n^2$ by 4 orthogonal array which is invariant under the Klein 4-group $K_4$ for all positive integers $n$ congruent to 0 modulo 4 except possibly for $n \in \{12, 24, 156, 348\}$.

0. Introduction

This note uses the terminology and notation of [2]. In what follows an orthogonal array will always be an $n^2 \times 4$ orthogonal array. An orthogonal array which is invariant under conjugation by the Klein 4-group but by no larger subgroup of $S_4$ (the symmetric group on $\{1, 2, 3, 4\}$) is called a $K_4$ orthogonal array (KOA). In [1] it is shown that the spectrum for KOAs is precisely the set of all $n \equiv 0$ or 1 (mod 4) except 5 and possibly 12 and 21.

Now a bit of reflection reveals that the rows of a KOA must look like $(a, a, a, a)$, $(a, a, b, b)$, $(a, b, a, b)$, $(a, b, b, a)$, or $(a, b, c, d)$ where $a$, $b$, $c$ and $d$ are distinct. Hence, if $(P, B)$ is a KOA, then the orbits of $K_4$ acting on $B$ look like

\[
\{(a, a, a, a)\}, \\
\{(a, a, b, b), (b, b, a, a)\}, \\
\{(a, b, a, b), (b, a, b, a)\}, \\
\{(a, b, b, a), (b, a, a, b)\}, \text{ or} \\
\{(a, b, c, d), (b, a, d, c), (c, d, a, b), (d, c, b, a)\}.
\]
We will denote the orbit containing \((x, y, z, w)\) by \([[(x, y, z, w)]\). Two orbits 
\([[(x_1, y_1, z_1, w_1)]\) and 
\([[(x_2, y_2, z_2, w_2)]\) are disjoint provided \(\{x_1, y_1, z_1, w_1\}\) and 
\(\{x_2, y_2, z_2, w_2\}\) are disjoint. A parallel class of \(B\) is a collection of pairwise disjoint 
orbits which partition \(P\) and \(B\) is said to be resolvable provided \(B\) can be partitioned 
into parallel classes. In what follows we will abbreviate resolvable KOA to RKOA.

The purpose of this note is to construct an \(n^2 \times 4\) RKOA for every \(n = 0 \mod 4\) except for 4 cases.

1. Construction of RKOAs.

CONSTRUCTION 1.1. Let \((Q, \circ)\) be a self-orthogonal quasigroup of order \(v\) (which we will always take to be idempotent) having an orthogonal mate \((Q, \circ)\) which 
is commutative. We make no assumptions concerning \((Q, \circ)\) other than the fact that 
it is commutative (and orthogonal to \((Q, \circ)\) of course). Set \(S = Q \times \{1, 2, 3, 4\}\) and 
define a collection of rows \(R\) of \(S\) as follows:

1. For all \(a \neq b \in Q\),
   \(i\) \(\begin{align*} &\text{if } a \circ b \neq c = x, \\
&\text{then } ((a, 1), (b, 1), (c, 2), (c, 2)) \in R \iff \{b, c\} \subseteq \{a\}\end{align*}\)
   \(\text{and}
   \begin{align*} &\text{if } a \circ b \neq c = x, \\
&\text{then } ((a, 1), (b, 2), (c, 2), (c, 2)) \in R \iff \{b, c\} \subseteq \{a\}\end{align*}\)
   \(\text{and}
   \begin{align*} &\text{if } a \circ b \neq c = x, \\
&\text{then } ((a, 1), (b, 3), (c, 2), (c, 2)) \in R \iff \{b, c\} \subseteq \{a\}\end{align*}\)

2. Let \((S, G, B)\) be a resolvable transversal design with groups 
\(G = \{Q \times \{1\}, Q \times \{2\}, Q \times \{3\}, Q \times \{4\}\}\) and blocks \(B\) containing as one parallel class 
of blocks 
\(\pi = \{(a, 1), (a, 2), (a, 3), (a, 4)\} \text{ if } a \in Q\). Let \(\begin{align*} &\text{if } a \neq b \in Q\text{ and } \{a, b\} \subseteq \{1, 2, 3, 4\}, \\
&\text{if } a \neq b \in Q\text{ and } \{a, b\} \subseteq \{1, 2, 3, 4\}, \\
&\text{if } a \neq b \in Q\text{ and } \{a, b\} \subseteq \{1, 2, 3, 4\},
\end{align*}\)

3. \(\begin{align*} &\text{if } a \neq b \in Q\text{ and } \{a, b\} \subseteq \{1, 2, 3, 4\}, \\
&\text{if } a \neq b \in Q\text{ and } \{a, b\} \subseteq \{1, 2, 3, 4\}, \\
&\text{if } a \neq b \in Q\text{ and } \{a, b\} \subseteq \{1, 2, 3, 4\},
\end{align*}\)

4. \(\begin{align*} &\text{if } a \neq b \in Q\text{ and } \{a, b\} \subseteq \{1, 2, 3, 4\}, \\
&\text{if } a \neq b \in Q\text{ and } \{a, b\} \subseteq \{1, 2, 3, 4\}, \\
&\text{if } a \neq b \in Q\text{ and } \{a, b\} \subseteq \{1, 2, 3, 4\},
\end{align*}\)

It is immediate that the orthogonal array \((S, R)\) is invariant under conjugation by 
at least \(K_4\). To see that \((S, R)\) is not invariant under conjugation by any larger 
subgroup of \(S_4\) it suffices to show that \((S, R)\) is not invariant under conjugation 
by the alternating group \(A_4\). However this is obvious since if \(a \neq b\), then 
\((a, 1), (b, 1), (a, 2), (b, 2)\) \(\in R\) but \((b, 1), (a, 2), (a, 1), (b, a, 2)\) \(\notin R\). We now 
show that, in fact, \((S, R)\) is resolvable. So, let \(x \in Q\) and set \(X = \{a \mid a \circ a = x\}\) and 
\(Y = \{b \in Q \mid b \circ c = c \circ b = x\}\). Then of course \(X \cup Y\) is a partition of \(Q\), since \((Q, \circ)\) 
is commutative. (We remark that \(X\) might well be empty.) Now define three parallel 
classes \(\pi_1(x), \pi_2(x), \pi_3(x)\) in the following manner:

\(\begin{align*} &\pi_1(x) = \{(a, 1), (a, 2), (a, 3), (a, 4)\} \cup \{\text{orbits of type } (1-i) \text{ iff } \{b, c\} \subseteq Y\}, \\
&\pi_2(x) = \{(a, 1), (a, 3), (a, 4), (a, 2)\} \cup \{\text{orbits of type } (1-ii) \text{ iff } \{b, c\} \subseteq Y\}, \text{ and} \\
&\pi_3(x) = \{(a, 1), (a, 4), (a, 2), (a, 3)\} \cup \{\text{orbits of type } (1-iii) \text{ iff } \{b, c\} \subseteq Y\}.
\end{align*}\)