TECHNICAL NOTE

Family of Optimally Conditioned Quasi-Newton Updates for Unconstrained Optimization

Y. F. Hu¹ AND C. Storey²

Communicated by L. C. W. Dixon

Abstract. In this note, a general optimal conditioning problem for updates which satisfy the quasi-Newton equation is solved. The new solution is a family of updates which contains other known optimally conditioned updates but also includes new formulas of increased rank. A new factorization formula for the Broyden family and some preliminary numerical results are also given.

Key Words. Unconstrained optimization, quasi-Newton updates, optimal conditioning, rank n family.

1. Introduction

Consider the unconstrained optimization problem,

\[
\min_{x \in \mathbb{R}^n} f(x).
\]

Quasi-Newton methods (see, e.g., Ref. 1) decide the new search direction \( d^+ \) at each iteration by

\[
d^+ = -H^+g^+,
\]

where \( x^+ \) is the current point, \( g^+ = \nabla f(x^+) \), and \( H^+ \) is an approximation to \( \{\nabla^2 f(x^+)\}^{-1} \). The matrix \( H^+ \) satisfies the quasi-Newton condition

\[
H^+\gamma = \delta,
\]

¹Graduate Student, Department of Mathematical Sciences, Loughborough University of Technology, Loughborough, Leicestershire, England.
²Professor, Department of Mathematical Sciences, Loughborough University of Technology, Loughborough, Leicestershire, England.
where \( y = g^+ - g, \) \( \delta = x^+ - x, \) \( x \) is the previous point, and \( g = \nabla f(x) \). There is, of course, much freedom in the selection of \( H^+ \), even though the quasi-Newton condition is satisfied, and other desirable properties such as symmetry and positive definiteness can be imposed on \( H^+ \).

An important family of formulas, for deriving \( H^+ \) from the previous matrix \( H \) (\( H \) is said to be updated to \( H^+ \)) is the self-scaling Broyden family of updates, which has the form

\[
H^+_B(\phi, \zeta) = \zeta (H - H \gamma^T H \gamma^T H \gamma + \phi \gamma^T H \gamma \nu \nu^T) + \delta \delta^T / \delta^T \gamma,
\]

where the subscript \( B \) stands for Broyden and where

\[
v = \delta / \delta^T \gamma - H \gamma^T H \gamma,
\]
\[
\zeta > 0 \text{ is the scaling factor, and } \phi \text{ a free parameter. Clearly, } H^+_B(\phi, \zeta) \text{ is symmetric if } H \text{ is symmetric, and writing}
\]
\[
b = \gamma^T H \gamma / \delta \gamma, \quad h = \delta^T H^{-1} \delta / \delta^T \gamma,
\]
then provided \( \delta \gamma > 0 \) and \( H \) is positive definite, \( H^+_B(\phi, \zeta) \) is positive definite if

\[
\phi > \bar{\phi} = -1 / (bh - 1).
\]

Note that here \( \bar{\phi} < 0 \), since \( bh \geq 1 \). Three well-known members of the family are the BFGS, DFP, and symmetric rank-one updates, which are given by \( \phi = 1, \phi = 0, \) and \( \phi = 1 / (1 - b \zeta), \) respectively. When \( \phi = \bar{\phi}, \) \( H^+_B(\phi, \zeta) \) becomes singular.

It is interesting that some members of the Broyden family can be obtained by minimizing appropriate measures of change. For example, the BFGS update can be obtained by solving the problem

\[
\min_{H^+} \| W^{1/2} (H^+ - H) W^{1/2} \|_F,
\]

\[
\text{s.t. } H^+ \gamma = \delta,
H^+ \tau = H^+,
\]

where \( W \) is some symmetric, positive-definite weighting matrix satisfying \( W \delta = \gamma; \) see e.g., Ref. 1. The inverse of the BFGS update, \( B^+ = (H^+)^{-1}, \) also solves the problem (Ref. 2)

\[
\min_{B^+} \psi (H^{1/2} B^+ H^{1/2}),
\]

\[
\text{s.t. } B^+ \delta = \gamma,
B^+ \tau = B^+,
B^+ > 0,
\psi(A) = \text{trace}(A) - \log(\text{det}(A)).
\]