TECHNICAL NOTE

Global Optimization of a Quadratic Functional with Quadratic Equality Constraints

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Communicated by F. E. Udwadia

Abstract. In this paper, we investigate a constrained optimization problem with a quadratic cost functional and two quadratic equality constraints. While it is obvious that, for a nonempty constraint set, there exists a global minimum cost, a method to determine if a given local solution yields the global minimum cost has not been established. We develop a necessary and sufficient condition that will guarantee that solutions of the optimization problem yield the global minimum cost. This constrained optimization problem occurs naturally in the computation of the phase margin for multivariable control systems. Our results guarantee that numerical routines can be developed that will converge to the global solution for the phase margin.

Key Words. Quadratic functionals, quadratic equality constraints, global optimization.

1. Introduction

This paper is concerned with the global optimization of a quadratic functional with two quadratic equality constraints. Specifically, we solve the problem

(P1) \( \min x^TQx, \)
\( \text{s.t. } x^TRx = 1, \)
\( x^TSx = 1, \)

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where $R > 0$, $S \geq 0$, all three matrices $Q, R, S \in \mathbb{R}^{n \times n}$, and $x \in \mathbb{R}^n$. The constraints are assumed to be feasible, so the existence of a global minimum is a trivial consequence of the continuity of the functional and the compactness of the constraint set. We also assume that the constraints are regular, so that the standard first- and second-order necessary conditions for a constrained extremum are satisfied. In particular, the second-order necessary condition states that the Hessian associated with the problem is positive semidefinite on the tangent plane to the constraint set at the global minimum. We will prove that, at the global minimum, the Hessian is necessarily positive semidefinite on the entire space $\mathbb{R}^n$, and not just on the tangent plane. It is also rather easy to see that a positive semidefinite Hessian on $\mathbb{R}^n$ is also sufficient for the local extremum to be a global minimum of the constrained optimization problem. The significance of this fact is that it establishes a stopping criterion for numerical schemes to approximate the global minimum.

The optimization problem considered here occurs naturally in the computation of the phase margin for multivariable control systems. Specifically, the computation of the phase margin requires solving a family of problems of the following form:

\[
(P2) \quad \min \; w^*(L + L^*)w, \\
\text{s.t.} \; w^*w = 1, \\
\quad w^*L^*Lw = 1,
\]

where $L = L(j\omega) \in \mathbb{C}^{n \times n}$ is a system transfer function matrix evaluated at the frequency $\omega \in \mathbb{R}$ and $w \in \mathbb{C}^n$. To solve this problem, the complex structure can be removed by using complexification, so that $\mathbb{C}^n$ is viewed at $\mathbb{R}^{2n}$ (for a detailed review of this problem and the complexification process, see Refs. 1 and 2). This results in the following constrained optimization problem:

\[
(P3) \quad \min \; \hat{w}^T(\hat{L} + \hat{L}^T)\hat{w}, \\
\text{s.t.} \; \hat{w}^T\hat{w} = 1, \\
\quad \hat{w}^T\hat{L}^T\hat{L}\hat{w} = 1,
\]

where $\hat{L} \in \mathbb{R}^{2n \times 2n}$, $\hat{w} \in \mathbb{R}^{2n}$ are the complexified forms of $L(j\omega)$ and $w$. Typical numerical methods used to solve such problems may only converge to local solutions.

Similar problems have been treated by many researchers [see Luenberger (Ref. 3), Mehrotra and Sun (Ref. 4), and Zhang (Ref. 5), for example]. The distinguishing factor for our problem $(P1)$ is that the constraints are equalities rather than inequalities as treated in the afore-