Control of Systems to Sets and Their Interiors

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Abstract. For a closed target set $S \subset \mathbb{R}^n$ and a control system (formulated as a differential inclusion and defined near $S$), the present paper considers a sufficient condition for the property that every point near $S$ can be steered to $S$ in finite time by some trajectory of the system. Estimates are obtained revealing how fast some such trajectory is nearing the target. A strong form of this condition is shown to imply that every trajectory of the system hits the target. With a further assumption on the target set $S$, we also consider conditions that guarantee that some trajectories enter the interior of $S$.

Key Words. Weak and strong attainability, control systems, penetrative systems, minimum time function.

1. Introduction

This paper presents conditions on a control system and a target set $S$ that guarantee that any point near $S$ can be steered to $S$ in finite time by at least one trajectory of the system. We call such a system and target set weakly attainable. Similar considerations are made for strongly attainable systems, in which all trajectories hit the target. Furthermore, in a similar vein, we consider conditions which allow for arcs to penetrate into the interior of $S$.

Our results are local in nature, and thus the target set $S \subset \mathbb{R}^n$ is assumed to be compact. We shall formulate the control system as a differential
inclusion,
\[ x(t) \in F(x(t)), \quad \text{a.e. } t \in [0, T], \]
\[ x(0) = x. \] (1)

in which \( F: \mathbb{R}^n \to \mathbb{R}^n \) is a multifunction with compact convex values. A sufficient condition for weak attainability (see Theorem 3.1 below) is that \( F \) be continuous and that there exists \( \delta > 0 \) so that, for every \( y \in \partial S \) and \( \zeta \in N_S^p(y) \), we have
\[ \min_{v \in F(y)} \langle \zeta, v \rangle \leq -\delta \| \zeta \|. \] (2)

For strong attainability, the min in (2) is replaced by max. The notation \( N_S^p(y) \) refers to the proximal normal cone at \( y \) (see Section 2). Notice that condition (2), the type of which we refer to as a \( \delta \) condition, is imposed only at points on the boundary of \( S \), and is nonvacuous at only those points that have a nonzero proximal normal. Condition (2) reduces to the one introduced by Petrov (Ref. 1), with \( S \) = the origin, but which is stated to hold in a neighborhood of the target. More recently, Bardi and Falcone (Ref. 2) and Cannarsa and Sinestrari (Ref. 3) have considered such local assumptions with general closed target sets, but the dynamics requires at least Lipschitz state dependence. We show in Theorem 3.4 that our continuity assumption on \( F \) can be further weakened to upper semicontinuity if the \( \delta \)-condition (2) is assumed to hold for all \( y \) near \( S \) and for any \( \zeta \in y - \text{proj}_S(y) \), where \( \text{proj}_S(y) \) are those elements in \( S \) closest to \( y \). Both continuity and upper semicontinuity are new in this context.

The proofs of our attainability results are based upon a time discretization algorithm, which can be described accurately as proximal projection. This in turn consists of applying the mean-value inequality (Ref. 4) in conjunction with new results in (Ref. 5) on properties of the distance function to a closed set. Our essentially algorithmic approach appears to yield the best rate of convergence among the existing ones.

Dynamic programming techniques are used in Ref. 2 to perform an approximation to the minimum time function, which is defined at a point \( x \) as the least time \( T \) for which a solution \( x(\cdot) \) to (1) exists and has \( x(T) \in S \). More specifically, a discretization of the associated Hamilton–Jacobi equation is made in Ref. 2, and then it is shown that the discrete solutions converge to the so-called viscosity solution. The discretization is in fact an approximation to the minimum time function because the latter is the viscosity solution (which is unique) of the Hamilton–Jacobi equation. A major difference with proximal projection as developed here from a method based on dynamic programming is that the former is moving forward in time, while the latter uses a backward construction. The appendix in Ref. 3 also