Diewert–Crouzeix Conjugation for General Quasiconvex Duality and Applications

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Abstract. A complicated factor in quasiconvex duality is the appearance of extra parameters. In order to avoid these extra parameters, one often has to restrict the class of quasiconvex functions. In this paper, by using the Diewert–Crouzeix conjugation, we present a duality without an extra parameter for general quasiconvex minimization problem. As an application, we prove a decentralization by prices for the Von Neumann equilibrium problem.

Key Words. Quasiconvex functions, duality.

1. Introduction

In an optimization problem, we have to minimize a function $f: \mathbb{R}^n \to \mathbb{R} (\mathbb{R} = \mathbb{R} \cup \{\pm \infty\})$ on a feasible set $X \subseteq \mathbb{R}^n$. The dual space is usually understood as the space of linear functions. By the inner product $\langle \cdot, \cdot \rangle$, each vector $x \in \mathbb{R}^n$ is associated with a linear function $\langle x, \cdot \rangle$. Therefore, we can in principle interpret the problem $\min \{ f(x): x \in X \}$ in the dual space. An interpretation in the dual space will be called a dual problem. With variations of the separation theorem (or Hahn–Banach theorem), the duality is a powerful instrument to interpret convexity. Over the last four decades, duality theory has been fully developed for convex minimization problems, where both the objective function $f$ and the feasible set $X$ are convex.

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More recently, with motivation from many applications, the convex-type duality has been extended to quasiconvex minimization problems (e.g., Refs. 1–28), where $f$ is a quasiconvex function and $X$ is a convex set. The epigraph of a quasiconvex function is no longer convex. This causes a fundamental difference between convex and quasiconvex duality. If a function is proper, closed, and convex, then it is a pointwise supremum of a family of affine functions and, by the Fenchel–Moreau theorem, we can obtain its dual representation. A crucial point in quasiconvex duality is how to obtain a dual representation for a quasiconvex function. For general quasiconvex functions, we have to use extra parameters to obtain dual representations (e.g., Refs. 1, 2, 4, 11, 15, 16, 18, 19, 21–24). If we want to avoid the extra parameters, then we often need to restrict the class of quasiconvex functions (e.g., Refs. 4, 7, 8, 17, 27, 28).

In this paper we use the conjugation given by Diewert (Ref. 8) and Crouzeix (Ref. 4) to present a duality without an extra parameter for general quasiconvex minimization problems. This conjugation originally was introduced in the study of the duality between direct and indirect utility functions (Refs. 4, 7, 8, 17), and now it is used to obtain a duality for more general problems. As an application, we prove a decentralization by prices in the Von Neumann equilibrium problem (Ref. 29).

The paper consists of five sections. We present general properties of the Diewert–Crouzeix conjugation in Section 2 and a quasiconvex duality in Section 3. We show a decentralization by prices for a class of quasiconvex minimization problems in Section 4; finally, we devote Section 5 to discussions and connections to previous results.

2. Diewert–Crouzeix Conjugation

In convex duality, the well-known Fenchel conjugate gives us a dual representation of a proper closed convex function. If we identify a function with its epigraph, then the Fenchel conjugate corresponds to a particular type of polarity (called upper conjugate sets in Ref. 30). In quasiconvex duality, we have to use other types of conjugation, because the epigraph of a quasiconvex function is no longer convex. For a general quasiconvex function, one often uses conjugations involving an additional parameter. If we restrict quasiconvex functions to a class of functions whose minimizer on the whole space is the origin, then we can use a symmetric quasiconjugation without extra parameters (Refs. 27 and 28). In this section, we use the Diewert–Crouzeix conjugation for another class of quasiconvex functions, whose maximizer on the whole space is the origin. First of all, let us give a motivation for the study on this subclass of quasiconvex functions.