Quasimonotone Variational Inequalities in Banach Spaces

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Abstract. Various existence results for variational inequalities in Banach spaces are derived, extending some recent results by Cottle and Yao. Generalized monotonicity as well as continuity assumptions on the operator f are weakened and, in some results, the regularity assumptions on the domain of f are relaxed significantly. The concept of inner point for subsets of Banach spaces proves to be useful.

Key Words. Variational inequality problems, quasimonotone operators, generalized monotone operators, inner points.

1. Introduction

In this article, we study the variational inequality problem which is defined as follows. Let B be a real Banach space, and let B* be its dual. The duality pairing between x∈B and u∈B* will be denoted by (x, u). Let K be a nonempty closed and convex subset of B, and let f: K→B*. The classical variational inequality problem (VIP) is to find x∈K such that

\[(x - \bar{x}, f(\bar{x})) \geq 0, \quad \text{for all } x \in K.\]

This problem has been extensively studied, both in finite-dimensional and in infinite-dimensional spaces; e.g., see Refs. 1-3 and the references therein. For an excellent survey in the finite-dimensional case, see Ref. 4.

In 1966, Hartman and Stampacchia (Ref. 5) showed that a (VIP) has a solution if f is continuous, B is a finite-dimensional space, and K is compact.

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Meanwhile, many extensions of this classical existence result have been obtained (e.g., the references in Ref. 3).

Most of the extensions of the result by Hartman and Stampacchia to infinite-dimensional spaces assume monotonicity of the operator \( f \). In recent years, several generalizations of monotonicity have been proposed (Refs. 6–8), which emerged from the concept of a pseudomonotone operator introduced by Karamardian in Ref. 9. He considered the nonlinear complementarity problem (NCP), which in fact is equivalent to a (VIP) where \( K \) is a cone. Karamardian (Ref. 9) proved that the (NCP) has a solution if \( B \) is finite-dimensional, \( f \) is a continuous, pseudomonotone operator, \( K \) is a pointed, solid, closed, convex cone, and there exists \( x \in K \) such that \( f(x) \) is an interior point of \( K^* \), the dual cone of \( K \). An extension of this result to general (VIP) where \( K \) is not necessarily a cone was stated by Harker and Pang in Ref. 4; for the proof, see Ref. 10.

Cottle and Yao (Ref. 11) extended Karamardian's result to (NCP) in real Hilbert spaces, assuming \( f \) to be continuous on finite-dimensional subspaces. Yao (Ref. 3) proved extensions of these results to the more general case where \( B \) is a real, reflexive Banach space. In Ref. 12, Yao derived results for more general (VIP) involving multivalued operators, hereby extending results by Shih and Tan (Ref. 13) for the monotone case. In all these articles, pseudomonotonicity of \( f \) is assumed.

In Ref. 7, the concept of quasimonotonicity, which is weaker than pseudomonotonicity, was introduced. In case of gradient maps, quasimonotonicity corresponds to quasiconvexity of the underlying function, just as pseudomonotonicity corresponds to pseudoconvexity (Ref. 7). It is well known that quasiconvex functions form a much broader class than pseudoconvex functions (Ref. 14). Hence, the class of quasimonotone operators is significantly larger than the class of pseudomonotone operators.

In this paper, existence results for (VIP) in reflexive Banach spaces are established for the more general case of quasimonotone operators. Most of the arguments used in the proofs are significantly different from those in the pseudomonotone case. Furthermore, in some of the results of the present paper, not only the assumptions on the operator \( f \), but also on the set \( K \), have been considerably weakened. Instead of assuming the existence of a topologically interior point, it is shown that the existence of a so-called inner point of \( K \) is sufficient. This concept of an inner point is more general than the concept of a relative algebraic interior point. For instance, any closed, convex subset of a separable Banach space has inner points, even if it does not have any relative algebraic interior points.

The paper is organized as follows. The remainder of this section contains notation and definitions. In Section 2, inner points will be introduced and some of their properties derived. In Section 3, several existence theorems