On the Complete Solution of $\epsilon y'' = y^3$

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Abstract. In Ref. 1, the author claimed that the problem $\epsilon y'' = y^3$ is soluble only for a certain range of the parameter $\epsilon$. An analytic approach, as adopted in the following contribution, reveals that a unique solution exists for any positive value of $\epsilon$. The solution is given in closed form by means of Jacobian elliptic functions, which can be numerically computed very efficiently. In the limit $\epsilon \rightarrow 0^+$, the solutions exhibit boundary-layer behavior at both endpoints. An easily interpretable approximate solution for small $\epsilon$ is obtained using a three-variable approach.

Key Words. Nonlinear boundary-value problems, elliptic integrals, Jacobian elliptic functions, singular-perturbation problems, matched asymptotic expansions.

1. Introduction

Consider the two-point boundary-value problem (Ref. 1)

\[ \epsilon y''(x) = y^3(x), \quad 0 < x < 1, \quad (1) \]
\[ y(0) = 1, \quad y(1) = 2, \quad (2) \]

with a constant $\epsilon > 0$.

Similar problems arise as models for certain catalytic reactions in chemical engineering (cf. Ref. 2, pp. 144–162 for details). Their solutions are related to those of (1)–(2), except that boundary layer behavior occurs at only one endpoint, which is a consequence of the non-Dirichlet condition at the other endpoint (cf. Ref. 3).

Howes (Ref. 4) gave a brief account of (1)–(2) in connection with stability conditions in singular perturbation theory, while Roberts (Ref. 1)
took up a numerical approach. The purpose of this paper is to give the exact solution of (1)–(2) and construct a uniformly valid approximation to the exact solution as \( \epsilon \) tends towards zero.

## 2. Analytic Approach

Multiplying (1) by \( 2y'/\epsilon \), we have

\[
(y^2)' = ((y^4 + C)/2\epsilon)',
\]

thus obtaining the first integral

\[
|y'| = \sqrt{(y^4 + C)/2\epsilon},
\]

where \( C \geq -1 \) for real-valued solutions \( y \).

As for the sign of \( y' \), it can be easily shown that, if \( y'(0) \geq 0 \),

\[
y' = +\sqrt{(y^4 + C)/2\epsilon}, \quad 0 < x < 1,
\]

and that, if \( y'(0) < 0 \),

\[
y' = \begin{cases} 
-\sqrt{(y^4 + C)/2\epsilon}, & \text{for } 0 < x < \xi, \\
0, & \text{for exactly one } \xi \in (0, 1), \\
+\sqrt{(y^4 + C)/2\epsilon}, & \text{for } \xi < x < 1,
\end{cases}
\]

with \( C \) ranging in \((-1, 0)\) in the latter case.

First, we single out the special case \( C = 0 \) in (5). The solution is readily found to be

\[
y = y(x; \epsilon) = 2/(2 - x), \quad \epsilon = 2.
\]

Integrating (6) for any nonzero value of \( C \) leads to the occurrence of an elliptic integral. Solving for \( y \) is a rather lengthy procedure then, and we ask the reader to refer to Ref. 3 for details.

It turns out that, for each value of \( \epsilon \), exactly one solution \( y(x; \epsilon) \) exists with either (5) or (6) applying for a certain value of \( C \). Table 1 summarizes the results obtained. Here, \( cn \) denotes the Jacobian elliptic function cosine amplitude with argument \( u \) as indicated and modulus \( k = 1/\sqrt{2} \) independent of \( \epsilon \); \( cn^{-1} \) specifies the inverse. The parameter \( \gamma = \sqrt{|C|} \) has been introduced for the sake of readability.

Note that

\[
\bar{\epsilon} = 1/(cn^{-1}(1/2, k))^2 = 0.76619889\ldots
\]

acts like a switch between solutions with positive initial slope only (entries 1–3 except for \( \bar{\epsilon} \)) and solutions with negative initial slope (entry 4), with \( y'(0; \bar{\epsilon}) = 0 \).