Lower Semicontinuous Solutions of the Bellman Equation for the Minimum Time Problem

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Abstract. The minimum time problem associated with a nonlinear control system is considered, and the unicity of the lower semicontinuous solution of the corresponding Bellman equation is investigated. A main tool in our approach is the Kruzhkov transformation that enables us to work on $\mathbb{R}^n - \{0\}$, where $\{0\}$ is the target set, instead of the unknown reachable set.

Key Words. Minimum time problems, Hamilton-Jacobi-Bellman equation, lower semicontinuous viscosity solutions.

1. Introduction and Main Results

Consider the nonlinear control system in $\mathbb{R}^n$

$$\begin{align*}
y'(t) &= f(y(t), u(t)), \quad t > 0, \\
y(0) &= x,
\end{align*}$$

(1a)

(1b)

where $u(t) \in U$, $U$ is a compact subset of $\mathbb{R}^m$, and $f: \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfies the following condition:

(H) $f(\cdot, \cdot)$ is continuous and there exists $L > 0$ such that $\|f(x, u) - f(y, u)\| \leq L \|x - y\|$, for all $x, y \in \mathbb{R}^n$ and $u \in U$; $f(y, U)$ is convex for all $y \in \mathbb{R}^n$.

The problem is that of reaching the origin $0$ of $\mathbb{R}^n$ (the target set) in minimum time, starting from the initial point $x \in \mathbb{R}^n$. The corresponding
Hamilton–Jacobi–Bellman equation is
\[
\max_{u \in U} \langle DV(x), -f(x, u) \rangle - 1 = 0, \quad \text{in } \mathbb{R}^n \setminus \{0\},
\]
where \(\mathcal{R}\) is the reachable set.

The literature on the minimum time function \(T(\cdot, \cdot)\), considered as a continuous solution of Eq. (2), is very rich (see Refs. 1–6 and the references therein. In the last years, problems have been investigated where the minimum time function is not continuous (see, e.g., Refs. 7–8). Among the above problems, we are interested in that associating a boundary condition to Eq. (2) in order to obtain the minimum time function as the unique solution.

As in Refs. 1, 2, 7, 8, we shall use a change of unknown variable in Eq. (2),
\[
W(x) = \begin{cases} 
1 - \exp(-V(x)), & x \in \mathcal{R}, \\
1, & \text{otherwise;}
\end{cases}
\]
this is the Kruzkov transformation; in this way, it is easy to recover both the reachable set and the minimum time function. Therefore, we are concerned with the equation
\[
\max_{u \in U} \langle DW(x), -f(x, u) \rangle + W(x) - 1 = 0, \quad \text{in } \mathbb{R}^n \setminus \{0\}. \tag{3}
\]

It should be observed that, under the hypotheses stated above, the minimum time function associated with (1) is lower semicontinuous (this follows from Ref. 9, p. 107) and
\[
\liminf_{x \to 0} T(x) = 0.
\]
If \(T^*\) is the Kruzkov transformation of \(T\), then \(T^*\) is also lower semicontinuous and
\[
\liminf_{x \to 0} T^*(x) = 0.
\]
Therefore, we set the boundary condition
\[
\liminf_{x \to 0} W(x) = 0. \tag{4}
\]
A solution of Eq. (3) is understood in the sense of lower semicontinuous viscosity solution as defined by Barron and Jensen in Ref. 10. From now on, we call it a BJ-solution; see Definition 2.1 in Section 2.

The main result is as follows.