SOME PROPERTIES OF E-m SEMIGROUPS

A. Cherubini Spoletini and A. Varisco

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In this note we study some properties of E-m semigroups, recently defined by Nordahl in [3]. In particular we study connections with power joined and strongly reversible semigroups. We prove, among other things, that an E-m semigroup is a disjoint union of power joined semigroups.

The second part of the paper deals with case m = 2, which presents interesting properties.

For undefined terminology and notation the reader is referred to [2].

1. THE GENERAL CASE

DEFINITION 1.1. A semigroup S which satisfies the identity

\[(ab)^m = a^m b^m \quad (a, b \in S; m \geq 1)\]

is called an E-m semigroup (Nordahl, [3]).

DEFINITION 1.2. A semigroup S is called a power joined semigroup if for any \(a, b \in S\) there exist two positive integers \(h, k\) such that

\[a^h = b^k\]

(Tamura, [8]).

We intend to show that an E-m semigroup is a disjoint union of power joined semigroups. To this end we prove the following,

**Lemma 1.3.** If S is an E-m semigroup, for any \(a, b \in S\) and any integer \(h > 0\), the following holds:

\[b^2 (ab)^h = (ba)^hm b^2\]
Proof. We shall prove the statement by induction: when \( h = 1 \) we have
\[
(2) \quad b^m (ab)^m = (b^m a^m b^m m = b^m (ab)^m m = (ba)^m b^m m^2
\]
thus (1) holds when \( h = 1 \). Suppose that (1) holds for \( h = k \), some positive integer; then using (2) we shall show that (1) holds for \( h = k + 1 \), as follows:
\[
\begin{align*}
&b^{m^2} (ab)(k+1)^m = b^{m^2} (ab)^{km} (ab)^m = (ba)^{km} b^{m^2} (ab)^m = (ba)^{km} (ba)^{m^2} = \\
&\quad = (ba)^{(k+1)m^2}. \quad \text{Thus (1) is established.}
\end{align*}
\]

**Lemma 1.4.** If \( S \) is an \( E-m \) semigroup, for any \( a, b \in S \) and any integer \( \lambda \geq 0 \), it does result
\[
(3) \quad b^{m+\lambda} m(m-1) (ab)^{m+\lambda} (m-1) = (ba)^{m(m-1)} b^{m+\lambda} m(m-1)
\]
Proof. When \( \lambda = 0 \), we have
\[
b^m (ab)^m = ([b(ab)^m] m = (ba)^m b^m m,
\]
so (3) holds for \( \lambda = 0 \); suppose that (3) holds for \( \lambda = \mu \), some non negative integer; then, using (1) with \( h = m - 1 \), we obtain
\[
\begin{align*}
&b^{m+\mu+1} m(m-1) (ab)^{m+\mu+1} (m-1) = b^{m+\mu+1} m(m-1) (ab)^{m+\mu+1} (m-1) = \\
&\quad = b^{m+\mu+1} m(m-1) (ba)^{m+\mu+1} (m-1) = b^{m+\mu+1} m(m-1) (ba)^{m+\mu+1} (m-1) = \\
&\quad = (ba)^{m+\mu+1} m(m-1) b^{m+\mu+1} m(m-1) = (ba)^{m+\mu+1} m(m-1) b^{m+\mu+1} m(m-1).
\end{align*}
\]
Thus (3) holds for \( \lambda = \mu + 1 \), and Lemma 1.4 is proved.

**Lemma 1.5.** An \( E-m \) semigroup is also an \( E-n \) semigroup for every integer \( n = m+\lambda m(m-1) \ (\lambda \geq 0) \).

Proof. Let \( S \) be an \( E-m \) semigroup. The statement is true for \( \lambda = 0 \). Suppose that it holds for \( \lambda = \mu \), some non negative integer; then, using (3), we find
\[
\begin{align*}
&(ab)^{m+\mu+1} m(m-1) = (ab)^{m+\mu+1} m(m-1) (ab)^{m+\mu+1} m(m-1) = \\
&\quad = (ab)^{m+\mu+1} m(m-1) (ba)^{m+\mu+1} m(m-1) = (ab)^{m+\mu+1} m(m-1) (ba)^{m+\mu+1} m(m-1) = \\
&\quad = a^{m+\mu+1} m(m-1) b^{m+\mu+1} m(m-1) = a^{m+\mu+1} m(m-1) b^{m+\mu+1} m(m-1), \quad \text{so Lemma 1.5 is proved.}
\end{align*}
\]

**Remark 1.6.** For \( m = 2 \), Lemma 1.5 states that an \( E-2 \) semi-