THE MANIFOLD E(L, L_m, L_{m+1}) IN AN n-DIMENSIONAL
PROJECTIVE SPACE P_n (m > 2)

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In [1] the manifold E(0, n − m, m), an m-parametric manifold whose element is an (n − m)-plane
L_{n−m} passing through a given point, is investigated by the Cartan method of exterior forms [2] and its
inseparable companion method of moving frames (polyhedra) of submanifolds [3] and systems of submani-
folds [4].

In the present article we invoke the same methods to study the manifold E(L, L_m, L_{m+1}), which is
an m-parametric manifold whose element comprises n − m linear (m + 1)-dimensional subspaces L_{m+1}^{α}
(α = m + 1, . . . , n) passing through a given m-plane L_m in which a point L is specified, where L_m is tan-
gent to the surface S_m described by the point L.

In § 1 we construct a semicanonical frame of a manifold E(L, L_m, L_{m+1}) in the sense of [3] and [2],
by analogy with the semicanonical frame of the manifold E(0, n − m, m) in [1]. In §§ 2 and 3 we give a geo-
metric characterization of the elements of the semicanonical frame of the manifold E(L, L_m, L_{m+1}).
With the aid of certain results from [1], in § 4 we define and give a geometric characterization of a cer-
tain projectively invariant class of manifolds E(L, L_m, L_{m+1}).

In § 5 we use a manifold E(L, L_m, L_{m+1}) to construct a semicanonical frame of the surface S_m in an
n-dimensional projective space P_n for the case m + 2 < n < m(m + 3)/2, for which the construc-
tion of an invariant clothing of the surface is demonstrated in [5-8, 11]. In § 5 we also investigate a cer-
tain projectively invariant class of manifolds E(L, L_m, L_{m+1}). It is shown that for manifolds of this class the (m + 1)-
planes L_{m+1}^{α} are invariantly related to the surface S_m described by the point L for n and m satisfying the
above-indicated inequalities. It turns out here that the frame constructed analytically in § 1 and geo-
metrically in §§ 2 and 3 for a manifold E(L, L_m, L_{m+1}) is still semicanonical for the given special class of
such manifolds. Consequently, we obtain a semicanonical frame for any surface S_m in P_n [m + 2 < n < m(m
+ 3)/2].

The notation and terminology are consistent with the usage of [1, 5] and [9].

§ 1. ANALYTIC CONSTRUCTION OF A SEMICANONICAL
MOVING TRIHEDRON

In an n-dimensional projective space P_n we investigate a manifold E(L, L_m, L_{m+1}), which is an
m-parametric manifold whose element comprises n − m linear (m + 1)-dimensional linear subspaces L_{m+1}^{α}
(α = m + 1, . . . , n) passing through an m-plane L_m in which a point L is specified, where the m-plane L_m is
the tangent m-plane to the m-surface S_m described by the point L. We join to E(L, L_m, L_{m+1}) a certain
analytic frame consisting of analytic points A_0, A_1, . . . , A_n. We seek the derivation formulas for this moving
trihe dron in the form

\[ dA_i = \omega_i^k A_k \quad (i, k = 0, 1, \ldots, n), \]

where the \( \omega_i^k \) are differential forms satisfying the structure equations

\[ D\omega_i^k = [\omega_i^j \omega_j^p] \quad (i, j = 0, 1, \ldots, n) \]

and the relation

\[ \omega_i^0 = 0. \]

We place the point A_0 at the point L, the points A_1, . . . , A_m in the tangent m-plane L_m to the surface
S_m at the point A_0, and the points A_{m+1} in the (m + 1)-plane L_{m+1}^α such that L_{m+1}^α = (L_m, A_{m+1}). Then the forms

\[ \omega^\alpha = \omega^\alpha_0, \quad \omega^\beta = \omega^\beta_0, \quad \omega^\gamma = \omega^\gamma_0 \quad (\alpha, \beta, \gamma = m + 1, \ldots, n; m; \alpha, \beta, \gamma = 1, 2, \ldots, m) \]
are principal forms. Taking the forms $\omega^\alpha$ as the independent ones, we write the system of differential equations of the manifold $E(L_2, L_\alpha, L_\beta)$ in the form

$$\omega^\xi = 0,$$

$$\omega^\xi = \Lambda^\xi_{\alpha \beta} \omega^\alpha (\alpha \neq \beta).$$

Exterior differentiation of the system (4) using the Cartan lemma yields the system

$$\omega^\xi = \Lambda^\xi_{\alpha \beta} \omega^\alpha (\alpha \neq \beta).$$

We obtain the following by exterior differentiation of (5) and (6) and application of the Cartan lemma:

$$\nabla \Lambda^\xi_{\alpha \beta} \equiv d \Lambda^\xi_{\alpha \beta} + \Lambda^\xi_{\alpha \gamma} \omega^\gamma - \Lambda^\xi_{\gamma \alpha} \omega^\gamma = \Lambda^\xi_{\alpha \beta}, \quad \omega^\gamma,$$

$$\nabla \Lambda^\xi_{\alpha \beta} \equiv d \Lambda^\xi_{\alpha \beta} + \Lambda^\xi_{\beta \gamma} \omega^\gamma - \Lambda^\xi_{\gamma \beta} \omega^\gamma - \Lambda^\xi_{\beta \gamma} \omega^\gamma = \Lambda^\xi_{\alpha \beta}, \quad \omega^\gamma.$$

Here the quantities $\Lambda^\alpha_{\beta \gamma}$ are symmetric with respect to any pair of subscripts, and the quantities $\Lambda^\alpha_{\beta \gamma}$ are symmetric with respect to the last two subscripts.

We let

$$A^\alpha_{\beta \gamma} = \Lambda^\alpha_{\beta \gamma}.$$

Here, as usual, the sign ( ) denotes symmetrization, and the sign [ ] denotes alternation with respect to the appropriate subscripts. It follows from (9) that the quantities $A^\alpha_{\beta \gamma}$ are symmetric with respect to any pair of subscripts.

Making use of the relations (7), inasmuch as $\pi_\alpha^\beta = 0 (\alpha \neq \beta)$, we find from (9)

$$d A^\alpha_{\beta \gamma} = (2 \pi_\alpha^\beta - \pi_\alpha^\beta - \pi_\alpha^\beta - \pi_\alpha^\beta - \pi_\alpha^\beta - \pi_\alpha^\beta) A^\alpha_{\beta \gamma}$$

(without summation over $\hat{a}$).

By means of the relations (8) and (10) we make the following fixation in the customary manner:

$$\Lambda_{\beta \gamma} = 0, \quad \Lambda_{\beta \gamma} = 0, \quad \pi_\alpha^\beta = 0 (\alpha \neq \beta);$$

$$\Lambda_{\beta \gamma} = 0, \quad A_{\beta \gamma} = 0, \quad \pi_\alpha^\beta = 0, \quad \pi_\alpha^\beta = 0,$$

$$A^\alpha_{\beta \gamma} = m, \quad m (\pi_\alpha^\beta + \pi_\alpha^\beta) = 2 \pi_\alpha^\beta$$

(without summation over $\hat{a}$).

Hence it follows that we can let

$$\omega^\xi = \Lambda^\xi_{\alpha \beta} \omega^\alpha,$$

where the quantities $\Lambda^\xi_{\alpha \beta}$ by virtue of (8), satisfy the relations

$$\Lambda_{\gamma \alpha} + \Lambda_{\gamma \beta} + \Lambda_{\gamma \alpha} + \Lambda_{\gamma \beta} = \Lambda_{\gamma \alpha} + \Lambda_{\gamma \beta} = \Lambda_{\gamma \alpha} + \Lambda_{\gamma \beta} (\gamma \neq \beta),$$

$$\Lambda_{\alpha \beta} + \Lambda_{\beta \gamma} = 0 (\beta \neq \gamma), \quad \hat{a} \neq \beta.$$

Exterior differentiation of the system (12) using the Cartan lemma yields the system

$$\nabla \Lambda^\xi_{\beta \gamma} \equiv d \Lambda^\xi_{\beta \gamma} + \Lambda^\xi_{\beta \gamma} \omega^\gamma - \Lambda^\xi_{\alpha \gamma} \omega^\gamma = \Lambda^\xi_{\beta \gamma}, \quad \omega^\gamma,$$

$$\nabla \Lambda^\xi_{\beta \gamma} \equiv d \Lambda^\xi_{\beta \gamma} + \Lambda^\xi_{\beta \gamma} \omega^\gamma - \Lambda^\xi_{\alpha \gamma} \omega^\gamma = \Lambda^\xi_{\beta \gamma}, \quad \omega^\gamma.$$

(summed over $\alpha$). Here the quantities $\Lambda^\xi_{\beta \gamma}$ and $\Lambda^\xi_{\beta \gamma}$ are symmetric with respect to the last two subscripts.