The transition radiation from a particle that intersects the waveguide perpendicular to its axis is considered. The expressions are derived for the radiation fields and intensities. The example of a rectangular waveguide is used to investigate the properties of the radiation, and the conditions which determine the Vavilov–Cerenkov radiation spectrum are derived.

Transition radiation in waveguides has been considered in a number of papers (see, for example, [1, 2]) in which the features of this radiation were investigated for motion of the charge parallel to the waveguide axis. The experimental difficulties that develop here, which are associated with the narrowband nature of the wave transformers and the separation between the beam and the radiation, may be excluded for motion of the beam in a transverse direction relative to the waveguide axis. This problem is partially considered in [3]. Below a theory of this phenomenon is proposed for a regular waveguide filled with a dielectric having the constants ε.

Let the waveguide axis coincide with the z axis of a certain rectangular coordinate system, and let the charged particle having a charge q move uniformly and in a straight line parallel to the 0x axis, crossing the waveguide walls at the points M1(x1, y0, 0) and M2(x2, y0, 0) during its motion. The charge and current densities of the moving particle may be written in the form

\[ \rho = q \delta(x - vt) \delta(y - y_0) \delta(z), \]
\[ j_x = j q \delta(x - vt) \delta(y - y_0) \delta(z). \]

In order to find the radiation field we use the method of [4]. We use the components of the electric and magnetic vectors E_Z and H_Z which are longitudinal relative to the waveguide axis as the field potentials; these components can easily be shown to satisfy the following equations on the basis of the system of Maxwell equations:

\[ \Delta E_{\omega z} + \frac{\omega^2}{c^2} \varepsilon E_{\omega z} = \frac{4\pi}{c} \frac{\partial \rho}{\partial z}, \]
\[ \Delta H_{\omega z} + \frac{\omega^2}{c^2} \varepsilon H_{\omega z} = \frac{4\pi}{c} \frac{\partial j_x}{\partial y}. \]

Note that the field defined by the potentials E_{ωz} yields a TM wave, while H_{ωz} yields a TE wave. In accordance with this we shall seek the solutions of Eqs. (2) in the form of expansion in orthonormalized eigenfunctions of the first and second boundary value problems for the waveguide cross section:

\[ E_{\omega z} = \sum_{n=0}^{\infty} E_n(z) \psi_n(x, y), \]
\[ H_{\omega z} = \sum_{n=0}^{\infty} H_n(z) \phi_n(x, y). \]

The functions \( \psi_n(x, y) \) and \( \phi_n(x, y) \) satisfy the following equations and boundary conditions:

\[ \Delta \psi_n(x, y) + \frac{\lambda_n^2}{c^2} \psi_n(x, y) = 0 \quad \text{and} \quad \phi_n|x=0 = 0. \]

where $A_\perp$ is the transverse two-dimensional part of the Laplace operator; $\Sigma$ is the contour of the waveguide cross section. If we now expand the right sides of Eqs. (2) in the respective eigenfunctions $\psi_n(x, y)$ and $\psi_n(x, y)$ with simultaneous expansion of all quantities in the $z$ coordinate in the Fourier integral, it follows that after a number of simple mathematical operators $E_n(z)$ and $H_n(z)$ may be represented in the following form:

$$E_n(z) = \frac{-i q}{\pi \varepsilon_0} A_n \int \frac{\varphi e^{i \omega z}}{\mu^2 - \gamma_n^2} d\mu, \quad H_n(z) = \frac{-q}{\pi c} B_n \int \frac{e^{i \omega z}}{\mu^2 - \gamma_n^2},$$

where

$$A_n = \int \exp \left( -i \frac{\omega}{c} \xi \right) \psi_n(\xi, \eta) d\xi, \quad B_n = \int \exp \left( -i \frac{\omega}{c} \xi \right) \frac{\partial \psi_n(\xi, \eta)}{\partial \xi} d\xi,$$

and

$$\gamma_n = \sqrt{\omega^2 / c^2 (1 - \delta_n^2)}, \quad \gamma_n = \sqrt{\omega^2 / c^2 (1 - \delta_n^2)}$$

are the propagation constants of TM and TE waves, respectively. The choice of the contour $c$ is dictated by the radiation conditions according to which waves traveling toward the radiation source must not exist. Note that since the $z > 0$ and $z < 0$ directions are equally legitimate in this considered problem, it follows that the solution must be symmetrical with respect to the $z$ coordinate. Performing the integration in the plane of the compact variable $\mu$ in Eqs. (5), we obtain

$$E_n(z) = \frac{-i q}{\pi \varepsilon_0} A_n \exp \left( -i \gamma_n |z| \right) \text{sgn} z, \quad H_n(z) = \frac{-q}{\pi c} B_n \exp \left( -i \gamma_n |z| \right).$$

The remaining field components can be expressed in terms of $E_n(z)$ and $H_n(z)$ by means of the equations [4]

$$E_i(\omega) = \sum_n \gamma_n^2 \frac{\partial E_n(z)}{\partial z} \psi_n(x, y),$$

$$H_i(\omega) = \frac{i \omega}{c} \sum_n \gamma_n^2 E_n(z) \left[ z_0 \nabla \psi_n(x, y) \right]$$

for the TM-wave, and

$$E_i(\omega) = - \frac{i \omega}{c} \sum_n \gamma_n^2 H_n(z) \left[ z_0 \nabla \psi_n(x, y) \right],$$

$$H_i(\omega) = - \sum_n \gamma_n^2 \frac{\partial H_n(z)}{\partial z} \nabla \psi_n(x, y)$$

for the TE-wave (here $z_0$ is the unit vector of the $z$ axis).

The energy of the transition radiation can be found by means of the Poynting vector. Let us make use of the expression for the radiation energy [4] which yields the following results for the $n$-th mode:

$$S_{n}^{(TM)} = \frac{q^2}{\varepsilon_0^2} \text{Re} \int \frac{\gamma_n |A_n|^2 \omega}{\varepsilon(\omega)} d\omega,$$

$$S_{n}^{(TE)} = \frac{q^2}{c^2 \varepsilon_0^2} \text{Re} \int \frac{|B_n|^2 \omega}{\varepsilon(\omega)} d\omega.$$

The total transition-radiation energy for $z > 0$ or $z < 0$ can be expressed by the formula

$$S = S^{(TE)} + S^{(TM)} = \sum_n S_{n}^{(TM)} + \sum_n S_{n}^{(TE)},$$

where the summation is performed over the propagating modes. By virtue of the symmetry of the problem which was noted above, it is evident from Eqs. (9) and (10) that an identical energy is radiated in the $z > 0$ and $z < 0$ directions.

For $\varepsilon = 1$ the equations presented above describe the effect of pure transition radiation which develop as a result of the crossing of the waveguide walls by a charge at the points $M_1(x_1, y_0, 0)$ and $M_2(x_2, y_0, 0)$. However, if $\varepsilon > 1$, then for $\beta \Gamma \varepsilon > 1$ Vavilov-Cerenkov radiation may also develop along with transition.