A criterion for the smooth finite determinacy of germs of $C^\infty$-diffeomorphisms is obtained.

1°. The purpose of this paper is to prove that the finite determinacy of a germ of a
$C^\infty$-diffeomorphism is equivalent to its formally finite determinacy. All formally finitely
determined diffeomorphisms can be described in analogy with the formal vector fields [1].

We make use of the technique of [2] for the solving of the functional equations arising
at the proof of the fundamental results.

We recall that a $C^\infty$-map germ $F:(\mathbb{R}^n, 0)\to(\mathbb{R}^n, 0)$ is said to be $k$-determined if each other
germs $G$, whose $k$-jet is equal to the $k$-jet of $F$, is conjugate with $F$ in the class $C^\infty$. A germ
is said to be finitely determined if it is $k$-determined for some $k < \infty$, and $\omega$-determined if
it is $k$-determined for $k = \infty$. The concept of finite determinacy has an obvious formal ana-
logue.

Let $F(x) = Ax + f(x)$, $f(x) = O(x)$, so that $A = F'(0)$ is a linear approximation of the
germs $F$. We denote by $L_c$ the so-called central manifold of the linear approximation, i.e.,
the invariant subspace corresponding to the unitary part of the spectrum. There exists a
formal transformation $\Phi:(\mathbb{R}^n, 0)\to(\mathbb{R}^n, 0)$ such that $L_c$ is invariant with respect to $\Phi F^{-1}$ (here
$\Phi$ is the formal Taylor series at zero). Therefore, without loss of generality we can assume that
$L_c$ is invariant with respect to $\Phi$.

**THEOREM.** Assume that the restriction $\hat{F}/L_c$ is finitely determined. Then the germ $F$ is
$\omega$-determined.

**COROLLARY.** The $C^\infty$-map germ $F$ is finitely determined if and only if the formal mapping
$\hat{F}$ is finitely determined.

2°. From the finite determinacy of the restriction $\hat{F}_c = \hat{F}/L_c$ there follows that $\dim L_c < 2$.
Moreover, if $\dim L_c = 1$, then $\hat{F}_c \neq 0$. If, however, $\dim L_c = 2$, then $\text{spec} \Lambda_c = \{e^{\pm i1}\}$;
here $\mu$ is irrational and, in addition, $\hat{F}_c$ is either a quasiextension or a quasicontraction.
We denote by $L_\pm$ the invariant subspaces of the linear approximation $\Lambda$, corresponding to the
eigenvalues that are, respectively, greater, smaller, in modulus than one. From the quasi-
hyperbolicity of the restriction $\hat{F}/L_c$ there follows (see [2], p. 128) the existence of a $C^\infty$-
transformation $\Phi$ such that the subspaces $L_+ + L_c$, $L_- + L_c$ are invariant with respect to $\Phi F^{-1}$. Therefore, we assume that this invariance condition is satisfied for $F$.

Assume now that $G = F + g$, $\Phi = 0$. We have to prove that the germ $G$ is conjugate with
$F$ in the class $C^\infty$. There exists a transformation $\Phi(x) = x + \varphi(x)$, $\Phi = 0$ such that $L_+ + L_c$ are
invariant with respect to $\Phi G^{-1}$ (see [2], p. 128). Since, as before, the difference $F - \Phi G^{-1}$ has a zero Taylor series at $x = 0$, we can assume that $L_+ + L_c$ are invariant with re-
spect to $G$. Finally, since the restriction $F_c = F/L_c$ is $\omega$-determined, we can assume that
$F_c = G_c$.

By virtue of [3], for the proof of the conjugacy of $F$ and $G$ it is sufficient to prove
the existence of a $C^\infty$-transformation $\Phi$ such that the difference $F - \Phi^{-1}G_\Phi$ is planar on $L_c$
(i.e., it is equal to $L_c$ to zero together with all the derivatives). Making use of Whitney's
theorem on the restoration of a $C^\infty$-mapping from the values of its derivatives (in the given
case on $L_c$), it is sufficient, in turn, to prove the existence of a transformation $\Phi(x) =
x + \varphi(x)$, $\Phi = 0$, such that

*An analogue of the corresponding theorem for diffeomorphisms has been obtained recently and
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(1) \[ F(x) - G(x) = 0 \quad (k = 0, 1, 2, \ldots). \]

Since \( F_c = G_c \), we can set \( \varphi | L_c = 0 \). Then all Eqs. (1) \( k = 1, 2, \ldots \) are linear with respect to the derivatives \( \varphi^{(k)}(x) x \in L_c \), and have the form

\[ \Psi(F(x)) = Q(x) \Psi(x) + \gamma(x) (x \in L_c), \quad (2) \]

where \( Q(0) \) is a nonsingular matrix.

We prove the solvability of the equation (2) for any complex-valued \( C^\omega \)-mappings \( Q: L_c \to C^\omega, \gamma: L_c \to C, \gamma = 0, \det Q(0) \neq 0 \). This will prove also our theorem.

3°. Assume first that \( \dim L_c = 1 \). Then \( F_c(0) = \pm 1 \). We consider the case \( F_c(0) = 1 \) and we prove the existence of a \( C^\omega \)-mapping \( \Psi: \mathbb{R} \to \mathbb{R}^\omega \), planar at zero and satisfying (2) for \( x < 0 \). In a similar manner one can prove the existence of a solution of this equation for \( x > 0 \), planar at zero.

Let \( \Psi \) be the algebraic closure of the field of quotients of the ring \( C[[x]] \) of formal series of one variable.

**LEMMA 1.** Let \( \det Q(0) \neq 0 \). There exists a matrix \( \hat{T}(x) \), invertible over the field \( \Psi \) and such that the matrix \( \hat{Q}(x) = (\hat{T}(\hat{F}_c x))^{-1} \hat{Q}(x) \hat{T}(x) \) is triangular.

**Proof.** We make use of the analogue of this statement for differential equations [4].

It can be verified in a straightforward manner that the formal diffeomorphism \( \hat{F}(x, y) = (\hat{F}(x), \hat{Q}(x) y) \) is included in the formal flow \( \hat{F}(x, y) = (\hat{F}(x), \hat{A}(x, y). \) Then \( \hat{F} = \hat{F}_1, \hat{Q}(x) = \hat{A} \times (x)_t \). Let \( a(x) = \frac{dA}{dx} \mid_{t=0} \). According to [4], there exists a transformation \( \hat{T}(x) \), invertible over \( \Psi \), such that the matrix

\[ \hat{a}(x) = (\hat{T}(x))^{-1} a(x) \hat{T}(x) - v(x) \hat{T}(x), \quad v = \frac{d\hat{T}(x)}{dx} \mid_{t=0} \]

has Jordan form over \( \Psi \). But then the matrix \( \hat{A}(x) = (\hat{T}(\hat{F}_c x))^{-1} \hat{A}(x) \hat{T}(x) \) is triangular over \( \Psi \). Setting \( t = 1 \), we obtain the assertion of the lemma.

By virtue of Lemma 1 we can assume that the matrix \( Q \) in Eq. (2) has the form \( Q(x) = D(x) + N(x) + \tau(x) \), where \( \tau = 0 \), \( N \) is a niltriangular matrix, while \( D(x) = \operatorname{diag}(\mu_1(x), \ldots, \mu_m(x)) \). Here \( \mu_j(x) = \exp \lambda_j(x^q) \), where \( \lambda_j \) are polynomials while \( m_j > 1 \) are integers (see [4]).

Since we assume \( x > 0 \) and (2) is solved in planar mappings, it follows that transformations of the form \( \varphi(x) = T(x) \Psi(x) \) are admissible, \( T \) being a \( C^\omega \)-matrix for \( x \neq 0 \), which becomes smooth at zero after the substitution \( x + xq \) and multiplication by \( x^\lambda \) for some \( q, \lambda \in \mathbb{R} \).

**LEMMA 2.** There exists an admissible transformation which reduces the matrix \( Q \) to triangular form.

**Proof.** Assume, for the sake of definiteness, that \( F_c(x) = x(1 + axr + \ldots), a > 0 \). By a transformation of the form \( T(x) = \operatorname{diag}(1, x^n, \ldots, x^{n-1}) \) we can achieve that \( N(x) = o(x^q) \) for any previously given \( q > 0 \). It is sufficient to select \( q = r \).

We order the eigenvalues \( \mu_1, \ldots, \mu_m \) so that \( |\mu_1(x)| > \ldots > |\mu_m(x)| \) for small \( x > 0 \). We shall seek the transformation of the matrix \( Q \) to an upper triangular matrix in the form \( T(x) = E + \Psi(x) \), where \( \Psi \) is a lower niltriangular \( C^\omega \)-matrix, planar at zero. We set \( Q(\lambda) = \lambda \Psi(x) + x^\lambda \Psi(\lambda) \). Then for the elements \( \Psi_{kj}(x) \) we obtain the equations

\[ \Psi_{kj}(F_c x) = (\mu_k(x) \mu_j^{-1}(x)) \Psi_{kj}(x) + \mu_j^{-1}(x) x^t \sum_i G_{kj}(x) \Psi_{ij}(x) - \mu_j^{-1}(x) x^t \sum_i \Psi_{kj}(F_c x) n_{ij}(x) - \mu_j^{-1}(x) \tau_{kj}(x) (k > j). \quad (3) \]

The elements \( \Psi_{kj} \) are obtained successively for \( j = 1, 2, \ldots \) from these equations, taking into account the equalities \( \Psi_{kj}(0) = 0 (k < j), \Psi_{kj}(0) = 0 (k > j) \). The elements \( n_{kj} \) for \( k < j \) are determined from the equalities

\[ -\sum_i \Psi_{kj}(F_c x) n_{ij}(x) + x^t \sum_i G_{kj}(x) \Psi_{ij}(x) = n_{ki}(x) + \sum_j \Psi_{kj}(x) n_{ij}(x) + \tau_{kj}(x). \]

For the determination of \( \Psi_{kj} \) for a fixed \( j \) we set \( g(x) = (\Psi_{j+1,1} \times (x), \ldots, \Psi_{mj}(x)) \). Then (3) can be written in the form

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