THEORY OF MULTIPLE \( j \)-ELEMENTARY MATRIX-VALUED FUNCTIONS WITH A POLE AT THE BOUNDARY OF THE UNIT CIRCLE

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The theory of a matrix-valued function \( \omega(\zeta) \), \( j \)-elementary in the unit circle and having a pole at the point \( \zeta_0 \), \( |\zeta_0| = 1 \), on the boundary of the unit circle, is considered. The structure of \( \omega(\zeta) \) is determined, conditions for the splitting-off of \( \omega(\zeta) \) from an arbitrary matrix-valued function \( W(\zeta) \), \( j \)-expanding in \( |\zeta| < 1 \), are formed, a theorem on the parametrization of a \( j \)-elementary matrix-valued function \( \omega(\zeta) \) of full rank is proved, and a decomposition of \( \omega(\zeta) \) of full rank into the product of parametrized \( j \)-elementary factors of full rank with simple poles at the point \( \zeta_0 \) is found.

We consider an object of V. P. Potapov's \( J \)-theory of analytic matrix-valued functions, namely a matrix-valued function \( \omega(\zeta) \), \( j \)-expanding \( \left( j = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \right) \) in the unit circle \( \omega(\zeta)j^\ast(\zeta) = j > 0, |\zeta| = 1 \), \( j \)-unitary on its boundary \( \omega(\zeta)j^\ast(\zeta) = j, |\zeta| = 1 \), and having a unique pole of multiplicity \( n \) at the boundary point \( \zeta_0 \) \( (|\zeta_0| = 1) \).

The theory of such an object includes the following questions: the description of the structure of \( \omega(\zeta) \), the splitting-off of \( \omega(\zeta) \) from an arbitrary \( j \)-expanding matrix-valued function with a pole of corresponding multiplicity at the point \( \zeta_0 \), the parametrization of a matrix-valued function \( \omega(\zeta) \) of full rank, and decomposition into binomial \( j \)-elementary factors.

We note that multiple \( J \)-elementary matrix-valued functions with a pole inside the domain have been investigated by the author in [1, 2]. One has considered also the boundary case, namely: one has investigated the theory of a multiple matrix function \( A(z) \), \( J_2 \)-expanding \( \left( J_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \right) \) in the upper semiplane \( \text{Im} z > 0 \), \( J_2 \)-unitary on the real axis \( \text{Im} z = 0 \) and having a pole at the point [2]. The examination of this special case has been suggested by the investigation by the methods of the \( J \)-theory of the power moment problem, to which \( A(z) \) is organically connected.

Of course, all the statement regarding the matrix-valued function \( \omega(\zeta) \) can be obtained from the corresponding facts for \( A(z) \) if with the aid of a linear fractional transformation one passes from the upper semiplane \( \text{Im} z > 0 \) to the unit circle \( (|\zeta| < 1) \) and with the aid of the Cayley transform one passes from \( J_2 \)-expanding matrix-valued functions \( A(z) \) to \( j \)-expanding matrix-valued functions \( \omega(\zeta) \). However, such a passage leads usually to a regrouping of the defining parameters and this makes difficult the description of the object with the aid of these parameters.

Therefore, it is convenient to consider \( \omega(\zeta) \) as an independent object of the \( J \)-theory and then to connect with it the multiple boundary interpolation problem of the Schur boundary problem type.

1. The Splitting-Off Inequality. The Accompanying Identity. The Structure of a \( j \)-Elementary Matrix-Valued Function \( \omega(\zeta) \). Splitting-off. Theorem 1.1. The Splitting-Off Inequality. The Accompanying Identity. First we consider the restrictions imposed on an arbitrary \( j \)-expanding matrix-valued function \( W(\zeta) \) by its behavior in the neighborhood of a given pole \( \zeta_0 \) \( (|\zeta_0| = 1) \), namely by the coefficients of its expansion into a Laurent series.

\textbf{THEOREM 1.1.} A matrix-valued function \( W(\zeta) \), \( j \)-expanding in \( |\zeta| < 1 \), having in the neighborhood of the point \( \zeta_0 \) \( (|\zeta_0| = 1) \) the Laurent series expansion

\( W(\zeta) = \frac{C_n}{(\zeta - \zeta_0)^n} + \cdots + \frac{C_1}{\zeta - \zeta_0} + C_0 + C_1 (\zeta - \zeta_0) + \cdots + C_{n-1} (\zeta - \zeta_0)^{n-1} + \cdots, \quad (1) \)

and \( j \)-unitary in the neighborhood of the pole on the unit circumference \(|\zeta_0| = 1\), satisfies the inequality

\[
\left( \frac{A}{-b_\xi(\xi)} A_1^* \right) \begin{pmatrix} \frac{\omega(\xi) - \omega^*(\xi)}{1 - \xi_0^2} \end{pmatrix} \geq 0,
\]

where \( A = A_1 \cdot A_2 \cdot A_3; \)

\[
A_1 = \begin{pmatrix} C_n^* & C_{n-1}^* & \cdots & C_2^* & C_1^* \\ C_{n-1} & C_{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_2 & 0 & \cdots & 0 \\ C_1 & 0 & \cdots & 0 \\ \end{pmatrix}
\]

\[
A_2 = \begin{pmatrix} -\zeta_0^j & 0 & 0 & \cdots & 0 \\ z_0^j & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\zeta_0^j & -2z_0^j & \cdots & -\zeta_0^j \\ (-1)^n \zeta_0^{n+1} & (-1)^n \zeta_0^{n+2} & \cdots & (-1)^n \zeta_0^{2n-1} \\ \end{pmatrix}
\]

\[
A_3 = \begin{pmatrix} C_0 C_{n+1} & \cdots & C_n & 0 \\ C_1 C_n & \cdots & C_{n+1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_n C_1 & \cdots & C_0 & 0 \\ \end{pmatrix}
\]

\[
b_n(\xi) = \begin{pmatrix} \frac{1}{\xi^2 - \xi_0^2} \\ \vdots \\ \frac{1}{\xi^{n+1} - \xi_0^{n+1}} \end{pmatrix}
\]

**Proof.** For the \( n + 1 \) points \( z_1, z_2, \ldots, z_n, \xi \) from the neighborhood of the point \( \xi_0 \) we write the Schwarz-Pick inequality for matrix-valued functions \( W(\xi) \):

\[
\begin{pmatrix} \frac{W^*(z_2)}{W(z_2)} - \frac{W(\xi)}{1 - \xi^2} & \frac{W^*(z_2)}{W(z_2)} - \frac{W(\xi)}{1 - \xi^2} \\ \frac{W(z_2)}{W(z_2)} - \frac{W(\xi)}{1 - \xi^2} & \frac{W(z_2)}{W(z_2)} - \frac{W(\xi)}{1 - \xi^2} \end{pmatrix} \geq 0.
\]

We multiply (6) on the right by the matrix \( ST \):

\[
S = \begin{pmatrix} (z_1 - \xi_0)^n & \cdots & (z_n - \xi_0)^n \end{pmatrix},
\]

\[
T = \begin{pmatrix} \omega_n(\xi) & 0 & \cdots & 0 \\ \omega_{n-1}(\xi) & \omega_n(\xi) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1(\xi) & \cdots & \omega_n(\xi) & \omega_1(\xi) \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix},
\]

\( \Omega_{n-k+1}(z) = (z - z_k) \cdots (z - z_n) \) and on the right by \( T^*S^* \) and we take the limit as \( z_k \to \xi_0 \) (along nontangential trajectories).

The elements of the limit matrix are computed according to the formulas [1]:

\[
A_{kl} = \lim_{z_\ell^* \to \xi_0} \sum_{\ell=1}^n \sum_{p=1}^n \frac{1}{\psi_{n-k+1}^* (z_\ell)} \frac{\psi_{n-k+1}^* (z_\ell - z_k)}{\psi_{n-k+1}^* (z_\ell - z_k)} \frac{1}{1 - \xi_0^2} \frac{1}{(n-k)! (n-l)!} \frac{\partial^{2n-k-l}}{\partial \xi^2 \partial \eta^{n-l}} \Theta(\xi, \eta) \bigg|_{\xi = \xi_0, \eta = \xi_0},
\]

\[
(l, k = 1, 2, \ldots, n),
\]

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