PARSEVAL'S EQUALITY IN THE ABSTRACT INTERPOLATION PROBLEM AND THE COUPLING OF OPEN SYSTEMS. II

A. Ya. Kheifets

UDC 517.54+519.98

The constructions described in Sec. 1 are applied to the investigation of the abstract interpolation problem. The general solution of the problem is the characteristic function of an operator colligation, obtained by the closure of fixed colligation by means of an arbitrary colligation with definite exterior spaces. The complete integral representation of a nonnegative quadratic form is obtained by applying Parseval's equality, considered in Sec. 1.

2°. The Abstract Interpolation Problem. 1. The Formulation of the Problem. The formulation of the problem and its motivation have been presented in [4].† Here we recall it.

Let $L_1$ and $L_2$ be Hilbert spaces, let $X$ be a linear space, let $T$ be a linear operator in $X$, let $D$ be a nonnegative sesquilinear form on $X$, and let $E$ and $M$ be linear mappings from $X$ into $L_1$ and $L_2$, respectively, such that we have the fundamental identity (FI):

$$D(x, y) - D(Tx, Ty) = \langle Ex, Ey \rangle - \langle Mx, My \rangle$$

(2.1)

A function $w(\zeta)$, holomorphic in the unit circle and whose values are linear contracting operators from $L_1$ into $L_2$, is said to be a solution of the abstract interpolation problem if there exists a linear mapping $F: X \to H^w$ with the properties

$$\langle Fx, Fx \rangle \leq D(x, x)$$

(2.2a)

$$FTx = t \cdot Fx - \left[ \begin{array}{c} I_{L_1} \\ \omega \\ I_{L_2} \end{array} \right] \left[ \begin{array}{c} Mx \\ Ex \end{array} \right] , (|t| = 1).$$

(2.2b)

One has to prove the existence and to describe all the solution $w(\zeta)$ of the problem, as well as the corresponding mappings $F$.

We mention that in many important special cases, equality (2.2b) can be rewritten in a somewhat different form: assume that for some $\zeta$, $|\zeta| < 1$, there exists $(\zeta - T)^{-1}$; then for this $\zeta$ we have

$$\zeta: (Fx)_{\zeta}(\xi) = (w(\xi) E - M)(\xi - T)^{-1} x$$

(2.2'+)

if, however, for some $\zeta$, $|\zeta| < 1$, then for this $(I_x - \overline{\zeta} T)^{-1}$ we have

$$\zeta: (Fx)_{\zeta}(\xi) = \overline{\zeta} (E - w(\xi)^* M)(I_x - \overline{\zeta} T)^{-1} x$$

(2.2'−)

2. Colligation Associated with the Problem. In the same way as in [4], with the data of the problem $(X, T, D, E, M)$ we associate an open system $\alpha$. To each $x \in X$ we associate an antilinear functional on $X$ (which we shall denote by $Dx$) of the form

$$Dx(y) = D(x, y)$$

(2.3)

In the space $\{Dx\}$ we introduce the inner product

$$\langle Dr_1, Dx_2 \rangle = D(x_1, x_2)$$

(2.4)

For $H$ we take the completion of the space $\{Dx\}$ with respect to the inner product introduced. The fundamental identity allows us to define an isometry from $H^w \oplus L_1$ into $H^w \oplus L_2$:

$$\sqrt{[DTx]} = [Dx]$$

(2.5)

†The first part of the paper has been published in Vol. 49 of this collection. The list of references is given there.


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The domain of definition (dV) of the operator V is the closure in \( H^a \oplus L_1 \) of the form
\[ DTx \oplus Ex, \]
while the domain of definition (dV) is the closure in \( H^a \oplus L_2 \) of the vectors of the form \( Dx \oplus Mx \). By \( N_V \) and \( M_V \) we denote the defect subspaces of V:
\[
N_V = (H^a \oplus L_1) \ominus d_V, \quad M_V = (H^a \oplus L_2) \ominus d_V.
\]
(2.6)

The isometry V can be extended (see, for example, [1]) to a unitary colligation, namely: let \( N_1 \) and \( N_2 \) be second copies of the spaces \( N_V \) and \( M_V \), respectively, considered as separate spaces. We define an operator \( A^\alpha \), mapping \( H^a \oplus L_1 \oplus N_2 = d_V \oplus N_2 \oplus N_3 \) onto \( H^a \oplus L_2 \oplus N_1 = \Delta_V \oplus M_V \oplus N_1 \), in the following manner:
\[
A^\alpha|d_V = V; \quad A^\alpha|N_1 \quad \text{identity mapping of } N_V \text{ onto } N_1; \quad A^\alpha|N_2 \quad \text{identity mapping of } N_2 \text{ onto } M_V.
\]

Obviously, \( A^\alpha \) is unitary. The outer spaces of the colligation are: \( N^\alpha_1 = N_1 \ominus L_1, \quad N^\alpha_2 = N_1 \oplus L_1 \). Obviously, \( P_{N^\alpha}, A^\alpha|N_3 = 0 \).

We denote the characteristic function of the colligation \( \alpha \) by \( S \) and we partition the latter into blocks:
\[
S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}: N_1 \oplus L_1 \rightarrow N_1 \oplus L_1.
\]


We give now a construction (coinciding, basically, with the construction applied in [1, 4]) allowing us to obtain the solutions of the abstract interpolation problem, and then we show that in this manner one obtains all the solutions. It is based on the closure, described in Sec. 2, of the colligation \( \alpha \) by an arbitrary colligation \( \beta \) with \( N^\beta_1 = N_1, \quad N^\beta_2 = N_2 \).

Let \( \omega(\zeta) \) be the characteristic function of the colligation \( \beta \), and let \( \gamma \) be the colligation obtained by the closure of the colligation \( \alpha \) by means of the colligation \( \beta \). The characteristic function of the colligation \( \gamma \) is equal to
\[
\omega = s_{11} + s_{12} \omega(1 - s_{11} \omega)^{-1} s_{12}
\]
(2.7)

This turns out to be the solution of the problem.

We set
\[
Fx = G^\gamma Dx
\]
(2.8)

where \( G^\gamma \) is the Fourier representation connected with the simple part of the colligation \( \gamma \), described in Sec. 1.7 [formula (1.24)]. For the verification we need only the relation (2.2b). Directly from the definitions there follow the relations
\[
G^\alpha(\zeta) P_{H^a} (1 - \zeta P_{H^a} A^\alpha - S(\zeta) P_{N^\alpha});
\]
(2.9+)
\[
G^\alpha(\zeta) P_{H^a} (1 - \zeta P_{H^a} (A^\alpha)^*) = \bar{\zeta} (P_{N^\alpha} (A^\alpha)^* - S(\zeta)^* P_{N^\alpha}).
\]
(2.9–)

From these relations, with the use of (1.24) (for \( H^a = H^a \oplus 0 \), (1.21), and (1.21'), we obtain
\[
G^\gamma(\zeta) P_{H^a} (1 - \zeta P_{H^a} A^\alpha) = (\psi(\zeta) \omega(\zeta) P_{N_1} + P_{L_1}) A^\alpha - (\psi(\zeta) P_{N_1} + w(\zeta) P_{L_1})
\]
(2.10+)

and
\[
G^\gamma(\zeta) P_{H^a} (1 - \zeta P_{H^a} (A^\alpha)^*) = \bar{\zeta} (\psi(\zeta)^* \omega(\zeta)^* P_{N_1} + P_{L_1}) (A^\alpha)^* - \bar{\zeta} (\psi(\zeta)^* P_{N_1} + w(\zeta)^* P_{L_1}).
\]
(2.10–)

Considering (2.10+) on vectors from \( d_V \) of the form \( DTx \oplus Ex \) and (2.10–) on vectors from \( d_V \) of the form \( Dx \oplus Mx \), we obtain (2.2b). Thus, \( w \) is a solution.

4. Some More Relations. From the consideration of (2.10+) on vectors from \( N_V \) and \( N_2 \) and (2.10–) on vectors from \( M_V \) and \( N_1 \), we obtain some additional relations, which will be needed in the sequel:
\[
G^\gamma(\zeta) P_{H^a} (A^\alpha)^*|N_1 = \psi(\zeta) \omega(\zeta)^* - \omega(\zeta)^* s_{21}(0)^*.
\]
(2.11+)
\[
\zeta G^\gamma(\zeta) P_{H^a} A^\alpha|N_2 = \psi(\zeta)^* - s_{11}(0);
\]
(2.12+)
\[
G^\gamma(\zeta) P_{H^a} A^\alpha|N_2 = \bar{\zeta} (\psi(\zeta)^* \omega(\zeta)^* - \omega(\zeta)^* s_{11}(0));
\]
(2.11–)
\[
G^\gamma(\zeta) P_{H^a} (A^\alpha)^*|N_1 = \psi(\zeta)^* - s_{22}(0)^*.
\]
(2.12–)