ON THE FORMAL EXPANSION OF THE POSSIBILITIES
OF THE METHOD OF INTEGRAL TRANSFORMATIONS IN
INVESTIGATING LINEAR DISTRIBUTED SYSTEMS WITH
CONSTANT PARAMETERS

A. S. Alekseev and O. K. Laburkina

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The Green's functions for a series of boundary-value problems are formulated in transforms with respect to t by means of matrix functions that are introduced and allow the scalar dependence on the weighted differences of the coordinate argument to be isolated in them. It was possible in general form to carry out the procedure of the method of integral transformations in solving the integral equations for the function that generates periodic motion.

The use of an integral that enters into an integral equation as an additional Laplace transform in solving these equations (specifically, the problem of finding the initial conditions for a distributed dynamic system that generates periodic motion reduces to these equations) was advanced by Fock in 1924 [1], by Wiener and Hopf in 1931 [2], and was then extensively used by other authors [3]. However, heretofore this method has been limited to a consideration of the particular case of equations having a kernel (or a Green's function) that depends on the difference or sum of the arguments.

The transforms in t of the Green's function formulated below for the general case of a one-dimensional distributed dynamic system with constant parameters and various linear boundary conditions in general depend on the weighted eigenvalues of the matrices B(p) of the transformed difference equations of the arguments; this impedes the direct application of the methods of [1-3] in the solution of the corresponding integral equations.

Henceforth considering for simplicity only nonsingular matrices B(p) which have different eigenvalues λi(p) for any values of the complex variable p, we note that the representation of the functions of these matrices via their principal components [5] Zi(p) which are matrices of the same order and are idempotent elements [4] allows a substantial expansion of the field of application of the method indicated [6].

Note also that the case of the appearance of multiple roots of the equation |Eλ - B(p)| = 0 for certain values of t (this case is essential in finding periodic processes) may be obtained from the considered case by means of a transition in the limit.

1. Let us consider the simplest case of a one-dimensional system that is distributed on the segment 0 ≤ x ≤ a and has dynamics described by the quantities

\[ u(t, x) = (u_1, u_2, ..., u_n), \]

\[ v(t, x) = (v_1, v_2, ..., v_m) \equiv \left( u, \frac{\partial u}{\partial x}, ..., \frac{\partial^{n-1} u}{\partial x^{n-1}} \right) \]

by means of the system of equations

\[ u_t = \sum_{i=0}^{n} A_i \frac{\partial^{n-i} u}{\partial x^{n-i}} + F(t, x). \]

\[ F(t, x) \in L_d([0, T] \times [0, a]) \]  

for the initial conditions

\[ u(0, x) = u^0(x) \in L_d([0, a]) \]  

and certain boundary conditions of various types:

\[ a) \quad \psi_0(t, 0) = \psi_0^0(t), \quad \psi_1(t, a) = \psi_1^0(t) \in L_d[0, T], \]  
\[ b) \quad \psi_0(t, 0) + D_1 \psi_1(t, 0) = C_n^{-1} \psi_1^0(t) \equiv \psi_0^1(t) \in L_d[0, T], \]  
\[ c) \quad \psi_0(t, 0) = \psi_0^0(t) \in L_d[0, T], \quad \psi_1(t, 0) = \psi_1^1(t, a), \]  
\[ d) \quad \psi_0(t, 0) = \psi_0^0(t) \in L_d[0, T], \quad \psi_1(t, 0) = \psi_1^1(t, a), \]  

where

\[ \psi_0 = (\psi_1, \psi_2, \ldots, \psi_k), \quad \psi_1 = (\psi_{k+1}, \psi_{k+2}, \ldots, \psi_m), \]  
\[ \psi_1^1 = (\psi_1, \psi_2, \ldots, \psi_{m-k}), \quad 2k \geq m, \quad A_j(n \times n), \]  
\[ C_0(nk \times nk), \quad D_0((m-k)n \times (m-k)n), \]  
\[ D_1(kn \times (m-k)n), \quad C_1((m-k)n \times kn) \]  

are the matrices of the constant elements; \( \det A_m \neq 0, \) \( \det C_0 \neq 0, \) \( \det D_0 \neq 0, \) \( F(t, x), \) \( \psi_0^0(t), \) \( \psi_0^1(t), \) \( \psi_1^0(t), \) \( \psi_1^1(t), \) \( \psi_0^0(t), \) \( \psi_1^1(t), \) \( \psi_1^1(t) \).

are columns of functions that are periodic in \( t \) (with the period \( T \)) and have the corresponding dimensionali-

\( \text{ties (n x 1), (kn x 1), (kn x 1), ((m-k) x 1).} \)

2. In order to obtain an integral equation that reflects the periodic motion of the system (1.1)-(1.7) we carry out certain additional constructions.

After transition to the transforms [7] in \( t \) in this system we write it in the form of a system of first-

order equations in the coordinates

\[ \bar{\psi}_s(p, x) = B(p) \bar{\psi}(p, x) + \bar{g}(p, x), \]  

where the matrix

\[ B(p) = \begin{bmatrix} \Theta & E & \Theta & \cdots & \Theta \\ \Theta & \Theta & E & \cdots & \Theta \\ \Theta & \Theta & \Theta & \cdots & \Theta \\ \cdots \cdots \cdots \cdots \cdots \Theta \\ \cdots \cdots \cdots \cdots \cdots \Theta \\ -a_0 & -a_1 & -a_2 & \cdots & E \end{bmatrix} \]  

for \( a_0 = A_m^{-1}(A_0 - pE), \) \( a_j = A_m^{-1}A_j \) has the matrix form (with cells \( (n x n) \) instead of scalars) of a matrix which yields the characteristic polynomial with coefficients from its bottom row, while

\[ \bar{g}(p, x) = (\Theta, -A_m^{-1}(u^0(x) + \bar{F}(p, x))) \]  

is a \((mn x 1)\) column in which only the \( n \) last elements are nonzero.

The solution of Eq. (2.1) is expressed in the form

\[ \bar{\psi} = e^{ax\bar{\psi}^0} + \int_0^x e^{B(x-y)} \bar{g}(p, y) dy, \]

in which the unknown vector \( \bar{\psi}^0 \) must generally be determined from one of the groups of boundary conditions (1.4)-(1.6).

In order to exclude this vector in a form that is convenient for subsequent analysis we shall represent (for the constraints mentioned earlier on the spectrum \( \lambda_i(p), i = 1, 2, \ldots, mn \) of the matrix \( B(p) \)) the