ON THE STATISTICAL MOMENTS OF THE WAVE FIELD
IN THE PRESENCE OF CAUSTICS

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The diffraction of a scalar wave field by a screen causing random and regular phase modulation is considered. The regular phase advance causes focusing of the field (i.e., the development of foci or caustic surfaces). It is shown that for strong phase fluctuations geometric optics is suitable for calculating the statistical moments of the field even in focusing regions if the singularity of the moment on the caustic is integrable. This is valid specifically for the average intensity. In calculating the higher moments it is necessary to take diffraction effects into account. Estimates are obtained of the intensity fluctuations in the region of the caustic shadow.

Many problems in the theory of wave propagation under natural conditions are problems in the diffraction of a partially coherent field. As an example, one may indicate the propagation of optical and radio emission in the interplanetary medium, and in the atmospheres of stars or planets. Random inhomogeneities of the refractive index of the medium generate fluctuations of the parameters of the wave, while its regular (large-scale) variations may lead to the development of foci or caustics. These are special regions of the wave field where geometric optics is not applicable and diffraction effects with a spatial scale $l_d$ are substantial (in the neighborhood of the simple caustic, for example, $I_d = k^{-2/3} R_c^{1/3}$; $R_c$ is the radius of curvature of the caustic; $k$ is the wave number [1]).

For strong fluctuations the diffraction pattern is "smeared," and it may be assumed that geometric optics is again suitable for the average quantities. Actually, the infinite value of intensity $I(r) = u(r)u^*(r)$ on the caustic (at the focus) is associated with the vanishing of the cross section of the ray tube in which the field energy is transferred. When allowance is made for diffraction the ray tubes spread and again acquire a finite width. But such a spreading may be caused by the fluctuational spreading of the rays as a result of scattering in a turbulent medium or by a stochastic screen. Therefore, for fairly strong phase fluctuations the average intensity $\langle I(r) \rangle$, calculated according to the method of geometric optics, must remain finite throughout, including in focusing regions — at the focus or on the caustic.

It is easy to check this assumption and likewise to clarify the problem of the higher statistical moments of the field concerning which nothing can be said a priori in view of the absence of any clear ray representations for them.

1. Let us consider a wave beam that propagates in the z direction and has passed through a phase screen in the $z = 0$ plane. The phase of the field on the screen has the form

$$\Phi(r') = -k\zeta(r') + \varphi(r'),$$

(1)

where $k\zeta(r')$ is the regular component, while $\varphi(r')$ is a random quantity having a zero average.

We shall consider the simplest fronts which lead to field focusing. Let

$$\zeta(r') = \frac{x^2}{2R_0} + \alpha x^3$$

( provided $\alpha > 0$)

(the one-dimensional case). For $\alpha = 0$ the radiation would concentrate on the focal line $x = 0, z = R_0$ ($R_0$ is the focal distance). The second term describes aberrations of the focusing system which are large in


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natural "lenses" (the earth's atmosphere [3, 4], the solar corona [5], etc.). For $\beta = 3$ a smooth (simple) caustic is formed; for $\beta = 4$ a symmetrical caustic with sharpening is formed, etc. In general an odd integer $\beta$ corresponds to caustic surfaces which have symmetry relative to the $x = 0$ plane and have no shadow region in the focal plane $z = R_0$. For even integers $\beta$ the caustics are nonsymmetrical: the region $x \neq 0$, $\text{sgn} x = \text{sgn} \alpha$ is a geometric-shadow region.

A ray arriving at the observation point $r(x, z)$ can be described by the equation

$$x = x' + \frac{d\xi}{dx'} (z - r(x')).$$

In the absence of fluctuations the intensity distribution in the observation plane is determined by the energy conservation law

$$I(x, z) \frac{dx}{I_{0}(x', z = 0)} = I_{0}(x', z = 0) dx'.$$

Using (2), we obtain the following result from this for the case of a nonsymmetrical caustic:

$$I(x, R_0) = \begin{cases} \frac{2I_0}{(\beta-1)(\beta\alpha R_0)^{1/2} |x|^{(\beta-1)/2}}, & \text{sgn} x = - \text{sgn} \alpha \\
0, & \text{sgn} x = \text{sgn} \alpha \end{cases}$$

(3)

For $x \rightarrow 0$ this function has a power-law singularity.

Proceeding in accordance with [2], we assume that the effect of inhomogeneities of the screen (phase fluctuations) leads to random displacements of the caustic relative to its average position with a standard $R_0 \sqrt{\langle \theta^2 \rangle}$, where $\langle \theta^2 \rangle$ is the mean-square of the random slopes of the equiphase surface. Then the statistical moments of the intensity may be written as

$$\langle I^{p/2}(x, R_0) \rangle = \int_{-\infty}^{\infty} I^{p/2}(x - \xi, R_0) w_0(\xi) d\xi,$$

(4)

where $w_0(\xi)$ is the distribution function of the positions of the caustic. The order $p$ of the field moment can formally be treated as a continuous parameter. Due to "random walks" of the caustic the "shadow-light" boundary spreads, and therefore the singularity of the distribution $I^{p/2}(x, R_0)$ smooths out as a result of averaging, provided only that the integral

$$\langle I^{p/2}(x, R_0) \rangle = \frac{(2I_0)^{p/2}}{(3-1)^{p/2}(\beta\alpha R_0)^{p/2(3-1)}} \int_{0}^{\infty} \frac{w_0(t + x)}{t^{p(3-2p/2(3-1))}} dt \quad (x > 0)$$

converges. For this it is sufficient if the condition

$$p \frac{\beta - 2}{2(\beta - 1)} < 1, \quad \text{or} \quad p < \frac{\beta - 1}{\beta - 2}$$

is fulfilled.

For this estimate it is obvious that in a lens having no aberrations ($\beta = 2$) all of the statistical moments of the field at the focus may be calculated according to geometric optics. In the case of a simple caustic ($\beta = 3$) such a calculation is already possible only for moments no higher than third (specifically, for the average intensity $\langle |r| \rangle$) (Fig. 1).

2. Let us consider those conditions for which one may neglect diffraction effects (i.e., make their spatial scale $l_d$ go to zero) in calculating the second moments. The transfer of the correlation function of the complex field $u$,

*This is associated with one general property which solutions of equations of the wave type (i.e., Helmholtz, Schrodinger, etc., equations) have. With respect to diffraction problems it may be formulated as follows:

$$\int |u_r(x, z)|^2 dx \rightarrow \int |u_r(x, z)|^2 dx,$$

where $u_r(x, z)$ is the geometric-optics solution. This relationship establishes the transition to a ray description of the fields at any point in space [6].