Local Times on Rays for a Class of Planar Lévy Processes¹

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Let $X, Y$ be independent Lévy processes on the real line. Assume $X$ and $Y$ admit Lebesgue measure as a reference measure, that $P^0(X_t > 0) = c$ for all $t > 0$ (or the weaker condition $P^0(X, > 0) \to c$ as $t \to \infty$) and that $Y$ has a local time at points. We investigate the distribution of the local time $L_t$ of $(X, Y)$ on the positive $x$-axis. It turns out that, under the first hypothesis (which is in particular satisfied by planar Brownian motion), if $T$ is an independent exponential time, then the ratio of $L_T$ to $L_T$, the local time on the entire $x$ axis, is (generalized) arc-sine and independent of $T$, and $L_T$ has a Gamma distribution. We obtain then expressions for the distribution of $L_T$. In the case of Brownian motion, the formula involves parabolic cylinder functions. Under the weaker condition mentioned above, together with mild secondary hypotheses, we obtain an expression for the asymptotic distribution of $L_T$ for large $t$.

KEY WORDS: Arc-sine law; Lévy process; stable process; local time.

1. INTRODUCTION

Let $X, Y$ be independent Lévy processes with infinite lifetimes on the real line, and let $m, m^2$ denote Lebesgue measure on $\mathbb{R}, \mathbb{R}^2$ respectively. We assume that the underlying sample space is rich enough to admit at least one Poisson process independent of $X$ and $Y$, and we let $P$ denote the measure under which $X_0 = Y_0 = 0$. (We will also use $P^{x,y}$ for the law of $(X, Y)$ starting at $(x, y)$, and $P^x_X$ for the law of $X$ starting at $x$, and similarly for $Y$.) We impose the following blanket hypotheses.

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799
X and Y admit m as a reference measure. (That is, \( U(B) := \int_0^\infty P(X_t \in B) \, dt \) defines a measure with \( U \ll m \), and similarly for Y.)

\[ U := S \cdot P(X, B) \, dt \]

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We shall in the last sections impose in addition one or other of the following conditions on \( X \):

1. There exists \( c, 0 < c < 1 \), such that \( P(X_t > 0) = c \) for all \( t > 0 \)
2. There exists \( c, 0 < c < 1 \), such that \( P(X_t > 0) \to c \) as \( t \to \infty \)

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Let \( \ell_t \) denote local time of \( Y \) at 0, so that \( \ell_t \), can also be viewed as local time of \((X, Y)\) on the x-axis, and let \( \tau_t \), denote the process inverse to \( \ell_t \). Let \( L_T := \int_0^T 1_{\{X_t > 0\}} \, d\ell_s \), the local time of \((X, Y)\) on the positive x-axis. Given \( q > 0 \), consider an independent exponential time \( T \) with parameter \( q \). We prove that \( L_T \) is independent of \( X(\tau_t) \), and then, by using an arc-sine law [Ref. 1, Th. (2.7)] for the latter process, we derive in Eq. (3.5) an appropriate form of an arc-sine law for \( L_T/\ell_T \). We then obtain under certain conditions the exact or asymptotic laws of \( L_T \) and \( L_T \). For example, under (1.3), it turns out Eq. (3.3) that \( L_T \) has a Gamma density, and in the special case where \( Y \) is Brownian motion, we obtain Eq. (3.16) an expression for the density of \( L_T \), involving parabolic cylinder functions. In Section 4, we work under the weaker condition in (1.4), obtaining in Eq. (4.14) an asymptotic version of the preceding results.

2. PRELIMINARIES

We assume throughout this section only that \( X \) and \( Y \) are real Lévy processes satisfying (1.1) and (1.2). Since \( Y \) admits \( m \) as a reference measure, there exists for each \( q > 0 \) a unique Borel function \( u^q : \mathbb{R} \to [0, \infty] \) such that \( u^q(x, y) := u^q(y - x) \) is \( q \)-excessive in \( x \) for each \( y \) and

\[ P_Y \int_0^\infty e^{-qf(Y_t)} \, dt = \int u^q(x, y) f(y) \, dy \]

for each Borel function \( f \geq 0 \). (Here, \( dy := m(dy) \).) See Ref. 2, [VI.1], for example. We normalize the local time \( \ell_t \), at 0 for \( Y \) so that

\[ P^x Y \int_0^\infty e^{-q\ell_t} \, d\ell_t, = u^q(0, 0) = u^q(-x) \]

\[ P^x Y \int_0^\infty e^{-q\ell_t} \, d\ell_t, = u^q(x, 0) = u^q(-x) \]

\[ \exists \text{c, } 0 < c < 1, \text{ such that } P(X_t > 0) = c \text{ for all } t > 0 \]
\[ \exists \text{c, } 0 < c < 1, \text{ such that } P(X_t > 0) \to c \text{ as } t \to \infty \]

Let \( \tau_t \), denote the right continuous process inverse to \( \ell_t \), so that under \( P \), \( \tau_t \), is a subordinator. Define the exponent function \( \psi \) for \((\tau_t)\) by

\[ P e^{-q\tau_t} = e^{-\psi(q)}, \quad q \geq 0 \]