Limit Theorems for Compact Two-Point Homogeneous Spaces of Large Dimensions

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Let $\mathbb{K}$ be the field $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ of real dimension $v$. For each dimension $d \geq 2$, we study isotropic random walks $(Y_t)_{t \geq 0}$ on the projective space $\mathbb{P}^d(\mathbb{K})$ with natural metric $D$ where the random walk starts at some $x_0^d \in \mathbb{P}^d(\mathbb{K})$ with jumps at each step of a size depending on $d$. Then the random variables $X_t^d := \cos D(Y_t^d, x_0^d)$ form a Markov chain on $[-1, 1]$ whose transition probabilities are related to Jacobi convolutions on $[-1, 1]$. We prove that, for $d \to \infty$, the random variables $(vd/2)(X_t^d) + 1$ tend in distribution to a noncentral $\chi^2$-distribution where the noncentrality parameter depends on relations between the numbers of steps and the jump sizes. We also derive another limit theorem for $\mathbb{P}^d(\mathbb{K})$ as well as the $d$-spheres $S^d$ for $d \to \infty$.

KEY WORDS: Projective spaces; $d$-spheres; isotropic random walks; central limit theorem; noncentral $\chi^2$-distribution; orthogonal polynomials; hypergroups.

1. LIMIT THEOREMS FOR ISOTROPIC RANDOM WALKS ON COMPACT TWO-POINT HOMOGENEOUS SPACES

1.1. Isotropic Random Walks on Compact Two-Point Homogeneous Spaces

Let $(X, D)$ be a two-point homogeneous space, i.e., $X$ is a locally compact metric space having a locally compact group $G$ of isometries of $X$ such that for all $x, y, u, v \in X$ with $D(x, y) = D(u, v)$ there is some $g \in G$ with $g(x) = u$ and $g(y) = v$. Then the stabilizer $H$ of some fixed $x_0 \in X$ is a compact subgroup of $G$, the homogeneous space $G/H$ can be identified with $X$, and the

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double coset space $G//H$ with the space of all orbits when $H$ acts on $X$. Moreover, if $G//H$ carries the quotient topology, then $\varphi: G//H \to [0, \infty[$, $HgH \mapsto D(gH, H)$ establishes a natural homeomorphism between $G//H$ and the set $I := \varphi(G//H) \subset [0, \infty[$.

In this paper we are interested in compact connected two-point homogeneous spaces $X$. These spaces were classified by Wang. They are equal either to a $d$-sphere $S^d$ for $d \geq 1$ or to a projective space $\mathbb{P}^d(\mathbb{R})$, $\mathbb{P}^d(\mathbb{C})$, $\mathbb{P}^d(\mathbb{H})$ for $d \geq 2$ or to $\mathbb{P}^2(\mathbb{C})$. In these cases, the space $I$ is a compact interval, and we may assume without loss of generality after a suitable linear transformation that $I = [0, \pi]$ holds.

Now let $(Y_n)_{n \in \mathbb{N}}$ be an isotropic random walk on some two-point homogeneous space $X$ starting at time 0 at the point $x_0$. The isotropy condition means that the transition probabilities satisfy

$$P(Y_n \in A \mid Y_{n-1} = x) = P(Y_n \in g(A) \mid Y_{n-1} = g(x)) \quad (1.1)$$

for all $n \in \mathbb{N}$, $x \in X$, $g \in G$ and Borel sets $A \subset X$. It is now clear that the distributions of $Y_n$ are $H$-invariant, and that they may be recovered from the distributions of the $[-1, 1]$-valued random variables $X_n := \cos D(Y_n, x_0)$ $(n \in \mathbb{N})$. Moreover, the sequence $(X_n)_{n \geq 0}$ forms a Markov chain.

For fixed two-point homogeneous spaces $X$, such associated Markov chains on $I \subset [0, \infty[$ and limit theorems for them were studied by many authors; see, for instance, Bingham, Diaconis, Voit, Zeuner, and references cited there. Moreover, in Voit we derived the following central limit theorem for Markov chains $(X_n)_{n \in \mathbb{N}}$ which were coming from isotropic random walks on $d$-spheres $S^d$ for different dimensions $d$ with $d \to \infty$.

**Theorem 1.1.** For each dimension $d \geq 2$, fix a jump distance $t(d) \in [0, \pi]$ and a number $l(d) \in \mathbb{N}$ of steps having the following properties:

1. $\lim_{d \to \infty} d^p \cdot t(d) = 0$ for some constant $p > 0$;
2. $\lim_{d \to \infty} [l(d)(1 - \cos t(d)) - 1/2 \cdot \ln d] = b$ exists for some constant $b \in \mathbb{R} \cup \{\infty\}$.

Now, for each dimension $d$, we consider the isotropic random walk $(Y_i^d)_{i \geq 0}$ on the $d$-sphere $S^d$ starting at some fixed point $x_0^d \in S^d$ such that its transition probabilities are given by jumps of fixed size $t(d)$ at each step. We next define the $[-1, 1]$-valued random variables

$$X_i^d := \cos \angle (Y_i^d, x_0^d) = \cos D(Y_i^d, x_0^d) \quad (i \geq 0)$$