The Occupation Time of Brownian Motion in a Ball

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Let $W$ be a Wiener process of dimension $d \geq 3$, starting from 0, and let $X(t)$ be the total time spent by $W$ in the ball centered at 0 with radius $t$. We give an affirmative answer to a conjecture of Taylor and Tricot on the tail distribution of $X(t)$. Lévy's lower functions of $X(t)$ are characterized by an integral test.

KEY WORDS: Occupation time; Lévy's lower class; Wiener process.

1. INTRODUCTION

Let $\{W(u), -\infty < u < +\infty\}$ be a "two-sided" $d$-dimensional Wiener process ($d \geq 3$) with $W(0) = 0$. Since it is transient, the total time spent by $W$ in any given ball is (almost surely) finite. The study of occupation times, apart from its own interest and from close relation with local times, is motivated by that of fractal geometry (Hausdorff measure, packing measure,...) of the Brownian curve. See for example Taylor and Tricot and references therein. Let

$$R(u) = \|W(u)\|, \quad u \in \mathbb{R}$$

be the radial part of $W$ (the symbol "\| \|" denoting the usual Euclidean norm in $\mathbb{R}^d$). In this paper, we are interested in

$$X(t) = \int_{-\infty}^{\infty} \mathbb{1}_{\{R(u) < t\}} \, du, \quad (t \geq 0)$$
the total time of $W$ spent in the ball $\{x \in \mathbb{R}^d: \|x\| < t\}$. Let us mention that another process, which bears some relation with $X$, was investigated by Meyre and Werner (11) who were interested in the case of a cone instead of a ball.

In their study of the law of the iterated logarithm (LIL) for $X(t)$, Taylor and Tricot (16) proved that for any $\varepsilon > 0$, when $x > 0$ is sufficiently small,

$$\exp\left(-\frac{2+\varepsilon}{x}\right) \leq \mathbb{P}[X(1) \leq x] \leq \exp\left(-\frac{2-\varepsilon}{x}\right) \quad (1.1)$$

from which they obtained a Chung-type LIL:

$$\lim \inf \frac{\log \log(1/t)}{t^2} X(t) = 2, \quad \text{a.s.} \quad (1.2)$$

The estimate in Eq. (1.1) is sufficient for the proof of Eq. (1.2), but fails to provide further information on the lim inf behavior of $X(t)$. The following conjecture was raised by Taylor and Tricot (16).

**Conjecture 1.1.** (Taylor and Tricot (16)). There exist constants $K$ and $\beta$, whose values depend only on $d$, such that

$$\mathbb{P}[X(1) \leq x] \sim Ke^{-2\beta x}, \quad x \to 0$$

Here and in the sequel, we adopt the notation $a(x) \sim b(x)$ (resp. $a(x) \ll b(x)$, as $x \to x_0$) meaning by $\lim_{x \to x_0} a(x)/b(x) = 1$ (resp. $\lim_{x \to x_0} a(x)/b(x) = 0$). It is shown in Section 3 that this conjecture is true and the exact values of $K$ and $\beta$ are given.

**Theorem 1.1.** For any $d \geq 3$, we have

$$\mathbb{P}[X(1) \leq x] \sim \gamma(d) x^\beta e^{-2\beta x}, \quad x \to 0$$

where

$$\beta = \frac{7}{2} - d \quad \text{and} \quad \gamma(d) = \frac{(8\pi)^{1/2}}{(\Gamma(d/2 - 1))^2}$$

When we try to make the LIL in Eq. (1.2) more precise, we must expect the lim inf behavior of $X(t)$ to depend on the value of $d$. Our main result is the following integral test, which characterizes the lower functions of $X(t)$.