OPTIMUM FIRST AND SECOND ORDER EXTRAPOLATIONS OF SUCCESSIVE OVERRELAXATION TYPE METHODS FOR CERTAIN CLASSES OF MATRICES

S. GALANIS, A. HADJIDIMOS and D. NOUTSOS

Department of Mathematics, University of Ioannina, GR-451 10 Ioannina, Greece
Department of Computer Sciences, Purdue University, West Lafayette, Indiana 47907, U.S.A.

Abstract.

This paper deals with the iterative solution of the linear system \( x = Bx + c \) when its Jacobi matrix \( B \) is weakly 2-cyclic consistently ordered and has a complex eigenvalue spectrum which lies on a straight-line segment. The optimization problem of the following three methods is considered and solved: i) The extrapolation of the optimum Successive Overrelaxation (SOR) ii) The second order extrapolation of a “good” SOR and iii) The second order extrapolation of the Gauss-Seidel method. In addition a variant of the second order methods considered, suitable for the solution of the system even if \( B \) is not necessarily weakly 2-cyclic consistently ordered, is proposed. Finally a reference to a theoretical comparison of the various optimum methods in the paper is made and their asymptotic convergence factors for selected eigenvalue spectra are illustrated in a Table in support of the theory developed.

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1. Introduction, notation and preliminaries.

Assume we are given the nonsingular linear system

\[
Ax = c,
\]

with \( A \in \mathbb{C}^{n \times n} \) and \( x, c \in \mathbb{C}^n \). Without loss of generality consider that \( A \) is written as

\[
A = I - L - U,
\]

where \( L \) and \( U \) are strictly lower and strictly upper triangular matrices and \( I \) is the unit matrix. Assume further that \( A \) is 2-cyclic consistently ordered (cf. [19], [20],...
and that its Jacobi matrix $B$

\begin{equation}
B := L + U
\end{equation}

in general possesses complex eigenvalues $\mu$ which lie on a straight-line segment (s.l.s.) such that

\begin{equation}
\mu \exp(-i\psi) \in [-\bar{\mu}, -\mu] \cup [\mu, \bar{\mu}] =: \mathcal{A},
\end{equation}

with

\begin{equation}
0 < \bar{\mu} < \mu (0 < \mu), \quad \psi \in [0, \pi), \quad \mu := \mu \exp(i\psi), \quad \bar{\mu} := \bar{\mu} \exp(i\psi)
\end{equation}

and $\mu < 1$ or $1 < \mu$ when $\psi = 0$.

Matrices $A$ possessing the above properties are very common in practice. They are usually obtained from the discretization of an elliptic partial differential equation by the finite difference method (cf. [19], [20], [4], [10]). These problems, in turn, arise in the theory of elasticity, fluid flow, neutron diffusion, steady-state electromagnetic field and heat flow, weather prediction etc. More specifically, consider the Helmholtz equation

\begin{equation}
-\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + cu = f, \quad (x_1, x_2) \in R := \{0 < x_i < l_i, \quad i = 1, 2\}
\end{equation}

\begin{equation}
u = g, \quad (x_1, x_2) \in \partial R := \{0 < x_i < l_i, \quad x_j = 0, l_j, \quad i, j = 1, 2, \quad i \neq j\},
\end{equation}

with $f$ and $g$ known functions and impose on $R \cup \partial R$ a uniform grid of mesh size $h_i = l_i/N_i$ in $x_i$-direction with $N_i$, $i = 1, 2$ integers. A discretization of (1.5) using central differences and a natural ordering of the nodes yields a linear system of the form (1.1) whose $(i_1, i_2)$th equation is given by

\begin{equation}
h_2 h_1^{-1} (u_{i_1-1, i_2} + u_{i_1+1, i_2}) + [2(h_1/h_2 + h_2/h_1) + h_1 h_2 c] u_{i_1, i_2}
\end{equation}

\begin{equation}
- h_1 h_2^{-1} (u_{i_1-1, i_2-1} + u_{i_1, i_2+1}) = h_1 h_2 f_{i_1, i_2}, \quad i_1 = 1(1)N_1, \quad j = 1, 2.
\end{equation}

For $c \geq 0$, as is the case with many practical problems, a premultiplication of the linear system (1.6) by $D^{-1}$, where the diagonal entries of the diagonal matrix $D$ are $2(h_1/h_2 + h_2/h_1) + h_1 h_2 c$, results in a 2-cyclic consistently ordered matrix $A$ of the form (1.2). Moreover the eigenvalues of $B$ in (1.3) are given by the expressions

\begin{equation}
\mu_{i_1, i_2} = 2\left(\frac{h_2}{h_1} \cos \frac{\pi i_1}{N_1} + \frac{h_1}{h_2} \cos \frac{\pi i_2}{N_2}\right) \left(2(h_1/h_2 + h_2/h_1) + h_1 h_2 c\right),
\end{equation}

\begin{equation}
i_1 = 1(1)N_1, \quad j = 1, 2.
\end{equation}

Obviously expressions (1.7) are real and satisfy the restrictions (1.4) with $\bar{\mu} < 1$ and $\psi = 0$. If at least one of $N_1$ and $N_2$ is odd then, except in some very special cases, $\mu > 0$. On the other hand a premultiplication of (1.6) by the $N_2 \times N_2$ block diagonal matrix $D^{-1}$, where the diagonal blocks of $D$ are the $N_1 \times N_1$ tridiagonal matrices