Homodyne Detection and Positive Operator-Valued Measures*

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We describe the homodyne detector in terms of positive operator-valued measures (POV-measures) and show that in the limit the homodyne detector reproduces the detector of quadrature phase amplitude of a signal field.

Introduction

The homodyne detector is a fundamental device for measuring phase-sensitive properties of optical fields. In this experimental arrangement the signal field beats against a local oscillator, which acts as phase reference.

In homodyne detection of a quasi-monochromatic input signal field of nominal frequency $\omega_0$, a partially transmitted beamsplitter with the transmittance $\varepsilon$ is used to mix the signal field with the strong local oscillator laser also of frequency $\omega_0$ in a coherent state $\vert \psi \rangle$ on the active region of a photodetector (Fig.1). The field modes are described by the annihilation operators $a$ and $b$.

In quantum optics it is usually assumed that such homodyne detector measures a variable called the quadrature phase (i.e. the observable proportional to $a + a^\dagger$) when the local oscillator amplitude is taken to infinity.

The first attempt to prove that the statistics of the homodyne detector can reproduce the detector of quadrature phase was made in [8, 9]. The proof was carried out by means of characteristic function. We will repeat this result in terms of POV-measures. This notion is more fundamental in quantum mechanics, whereas

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the characteristic function only have a meaning as calculation tools. Also the precise mathematical meaning of the limiting procedure will be given (\textit{*-weak p-convergence} of POV-measures). The asymptotic agreement between the homodyne detection and the quadrature phase observable was studied in [2].

\begin{figure}
\centering
\includegraphics[width=\textwidth]{homodyne_detector.png}
\caption{Homodyne detector.}
\end{figure}

The description of measured observables in quantum mechanics by means of POV-measures is old (see for example [3, 5]). Generally, POV-measures arise when we try to provide a more detailed description of an experimental setup and the whole measurement procedure than it is customarily done [4]. In turn, the generalization of the standard notion of an observable within the interpretational framework of quantum mechanics is the following.

Owing to the spectral theorem, an observable in quantum mechanics is usually represented either as a selfadjoint operator \( A \) or, equivalently, as a spectral measure \( E^A \). The properties of \( E^A \) guarantee the function \( p^A_\rho \) defined as

\begin{equation}
 p^A_\rho(\Delta) = \text{Tr}_\rho E^A(\Delta),
\end{equation}

where \( \Delta \) is a Borel subset of the real line and \( \rho \) is a density operator describing the state of the system, is a probability measure. A simple observation shows that the property of \( E^A \) being a projection is redundant. Any POV-measure preserves the sense of (1) as the probability measure and, as a consequence, the probabilistic Born interpretation of quantum mechanics.