Every Path Space of Dimension Two is Projectively Related to a Finsler Space

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The present paper is motivated by biological theory on time-sequencing changes along growth curves and the projective equivalence of Finsler physical theories. We extend classical work of G. Darboux (1894) in order to prove that any path space (i.e. local spray) of dimension two is projectively Finsler.

Introduction

Recently the so-called classical differential geometry of paths, originated by Princeton School in 1920's, has been considered to be more and more significant from the standpoint of applications [3]. In particular, the metrizability of path spaces (i.e. when paths are actually geodesics of a Riemannian or Finsler metric) is an essential problem in various fields of biology and physics [1 - 4].

The main purpose of the present paper is to show that the problem has positive solution if the dimension is 2 and time-sequencing parameter changes along all paths are allowed (i.e. projective change).

Theorem In two dimensions any path space is projectively related to a Finsler space.

We shall use the standard concept of projectively equivalent class of Finsler spaces, and show that an equivalence class of Finsler spaces of dimension two has the freedom of selection of two functions, each of two arguments.
1 The Metrizability Problem

L. P. Eisenhart, T. Y. Thomas and O. Veblen [6] were concerned with the metrizability problem of affine paths: given the equations of paths

\[ \frac{d^2x^i}{dt^2} + \Gamma^i_{jk}(z) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \]  \hspace{1cm} (1.1)

in a differentiable manifold \( M^n \) of \( n \) dimensions, find the condition for the existence of a Riemannian space whose geodesics coincide with the paths.

This problem may be also stated as follows: given a linear connection \( \Gamma = \{ \Gamma^i_{jk}(z) \} \) in \( M^n \) find a Riemannian metric tensor \( g_{ij}(x) \) on \( M^n \) such that \( \Gamma^i_{jk}(z) \) are the Christoffel symbols of the second kind constructed from \( g_{ij}(x) \). Thus, one is led to the differential equations

\[ \nabla_k g_{ij} = \partial_k g_{ij} - g_{rj} \Gamma^r_{ik} - g_{ir} \Gamma^r_{jk} = 0, \]  \hspace{1cm} (1.2)

for the Riemannian metrizability, where \( \Gamma^r_{jk}(z) \) are assumed to be symmetric in \( j \) and \( k \).

In 1954 O. Varga [11] considered the Finslerian metrizability problem based on the famous Cartan connection (the “Levi-Civita connection” of Finsler geometry). Given connection \( C \Gamma = \{ \Gamma^r_{ij}(x, y), C_{jk}(z, y) \} \) in the tangent bundle \( TM^n \) find a Finsler space \( F^n = (M^n, L(x, y)) \) such that \( C \Gamma \) coincides with the Cartan connection, this being determined from the fundamental tensor \( g_{ij}(x, y) = \delta_i \delta_j (L^2/2) \), where \( \delta_i \equiv \partial / \partial y^i \). Therefore, Varga was led to the system of differential equations similar to (1.2):

\[ \nabla^x g_{ij} = \partial_k g_{ij} - g_{rj} \Gamma^r_{ik} - g_{ir} \Gamma^r_{jk}, \]
\[ \nabla^y g_{ij} = \partial_k g_{ij} - g_{rj} C^r_{ik} - g_{ir} C^r_{jk}, \]  \hspace{1cm} (1.3)

for the Finslerian metrizability, where \( \partial_k = \partial_k - (y^r \Gamma^r_{ij}) \partial_r \).

Twenty years later, L. Tamássy [10] dealt with the problem above with \( g_{ij}(x, y) \) together with its reciprocal \( g^{ij}(x, y) \) but his result seems to us a little simpler than that of Varga. On the other hand, A. Rapcsák [9] considered the same problem in another form: given the equations of generalized paths

\[ \frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0, \]  \hspace{1cm} (1.4)

find a Finsler space \( F^n \) such that \( G^i_{jk}(x, y) = \partial_j \partial_k G^i(x, y) \) coincide with the connection coefficients of the Berwald connection, \( B \Gamma = \{ G^i_{jk}(x, y) \} \) constructed from the fundamental function \( L(x, y) \). Here, \( G^i \) is assumed positively homogeneous of degree two in \( y^j \).

It follows from Euler’s theorem on homogeneous functions that (1.4) can be written in the form

\[ \frac{d^2x^i}{dt^2} + G^i_{jk}(x, \frac{dx}{dt}) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0. \]  \hspace{1cm} (1.4')