E-Methods for Fixed Point Equations $f(x)=x$

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Abstract — Zusammenfassung

E-Methods for Fixed Point Equations $f(x)=x$. This paper provides newly implemented [11], [13] and widely applicable methods for computing inclusion (i.e. a containing interval) (Einschließung) of the solution of a fixed point equation $f(x)=x$ as well as automatic verification the existence (Existenz) and uniqueness (Eindeutigkeit) of the solution. These methods make essential use of a new computer arithmetic defined by semimorphisms as developed in [7] and [8]. We call such methods E-Methods in correspondence to the three German words. A priori estimations such as a bound for a Lipschitz constant etc. are not required by the new algorithm. So the algorithm including the a posteriori proof of existence and uniqueness of the fixed point is programmable on computers for linear as well as for nonlinear problems. This is a key feature of our results. The computations produced by E-methods deliver answers the components of which have accuracy better than $10^{-t+1}$ (where $t$ denotes the mantissa length employed in the computer).

Key words: E-method, inclusion, automatic verification.

AMS Subject Classification: 65H99.

1. Introduction

In [5], [10] and [11] methods are introduced, which provide an inclusion (i.e. a containing interval) of the fixed point of an equation. The methods derived in [5] are typically generalizations of those introduced by Moore in [9]. The results presented here both generalize and simplify the methods given in [5], [9] and [10].

The following iteration operator introduced in [6]

$$K(X) := \bar{x} - R \ast g(\bar{x}) + \{E - R \ast g'(X)\} \ast (X - \bar{x})$$

(1)
is used in [9]. Here \( \bar{x} \in \mathbb{R}^n \), \( g : \mathbb{R}^n \to \mathbb{R}^n \in \mathcal{C}_X^1 \) and \( X \in \mathbb{R}^n \) denotes an \( n \)-dimensional interval vector and \( E \) the \( n \times n \) identity matrix. In [9] \( R \) is required to be a real non-singular matrix and \( \bar{x} \in X \). Under these conditions the existence of a solution of \( g(\bar{x}) = 0 \) in \( X \) is derived from the property \( K(X) \subseteq X \). We will show that it is not necessary to assume \( R \) to be non-singular and that \( \bar{x} \) can be chosen arbitrarily (not necessarily \( \bar{x} \in X \)). The somewhat more stringent condition \( K(X) \subseteq X \) (which is almost always satisfied on the computer) is sufficient to show that \( R \) is non-singular and that \( g(x) = 0 \) has exactly one solution \( \bar{x} \in X \).

### 2. Theoretical Preliminaries

**Definition 1:** Let \( M_1 \) and \( M_2 \) be closed subsets of the locally convex topological space \( \mathcal{M} \). We define the strict inclusion relation as follows

\[
M_1 \prec M_2 : \Leftrightarrow M_1 \subseteq M_2.
\]

i.e., \( M_1 \) lies in the interior of \( M_2 \).

An improved form of some fundamental results of [5] and [10] is given in the following theorem.

**Theorem 2:** Let \( f : Y \to \mathcal{M} \) be a continuous mapping and \( F : \mathcal{P}\mathcal{M} \to \mathcal{P}\mathcal{M} \) an arbitrary mapping of the power set \( \mathcal{P}\mathcal{M} \) into itself such that

\[
x \in Y \Rightarrow f(x) \in F(Y).
\]

Let \( Y \) be convex and compact. If

\[
F(Y) \subseteq Y,
\]

then there exists a fixed point \( \hat{x} \) of \( f \) with

\[
\hat{x} \in F(Y) \subseteq Y.
\]

Moreover

\[
\hat{x} \in \bigcap_{i=0}^{\infty} F^i(Y),
\]

and \( Q(f, Y) \subseteq Y \) for the set of fixed points

\[
Q(f, Y) := \{ x \in Y | f(x) = x \}
\]

of \( f \) in \( Y \). Therefore \( Q(f, Y) \cap \partial Y = \emptyset \).

**Proof:** From (3) and (4) it follows immediately that \( f \) maps the convex and compact subset \( Y \) of the locally convex space \( \mathcal{M} \) into itself. According to the fixed point theorem of Schauder-Tychonoff \( f \) has at least one fixed point \( \hat{x} \in Y \). With \( \hat{x} \in Y = F^0(Y) \) we have by induction

\[
\hat{x} \in F^k(Y) \Rightarrow \hat{x} = f(\hat{x}) \in F(F^k(Y)) = F^{k+1}(Y).
\]

The proof of (7) derives from (3) and (4) since for all \( x \in Q(f, Y) \subseteq Y \) we have

\[
x \in Y \Rightarrow x = f(x) \in F(Y) \subseteq Y.
\]