On the Numerical Solution of Nonlinear Eigenvalue Problems

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Abstract — Zusammenfassung

On the Numerical Solution of Nonlinear Eigenvalue Problems. We consider the numerical solution of the nonlinear eigenvalue problem \( A(\lambda)x = 0 \), where the matrix \( A(\lambda) \) is dependent on the eigenvalue parameter \( \lambda \) nonlinearly. Some new methods (the BDS methods) are presented, together with the analysis of the condition of the methods. Numerical examples comparing the methods are given.

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Über die numerische Lösung nichtlinearer Eigenwertprobleme. Wir betrachten die numerische Lösung des nichtlinearen Eigenwertproblems \( A(\lambda)x = 0 \), wobei die Matrix \( A(\lambda) \) in nichtlinearer Weise vom Eigenwertparameter \( \lambda \) abhängt. Einige neue Methoden (die BDS Methoden) werden zusammen mit einer Untersuchung der Bedingungen dieser Methoden vorgestellt. Numerische Beispiele, welche diese Methoden vergleichen, werden präsentiert.

1. Introduction

Consider the nonlinear eigenvalue problem (NEVP) for the \( n \times n \) matrix \( A(\lambda, \rho) \), which is nonlinear in \( \lambda \) (the eigenvalue parameter) and \( \rho \) (a single real design parameter):

\[
A(\lambda(\rho), \rho)x(\rho) = 0, \quad y(\rho)^T A(\lambda(\rho), \rho) = 0^T,
\]

(1)
together with some desirable scaling schemes for the right and left eigenvectors \( x \) and \( y \). We denote the transpose and conjugate transpose by \((\cdot)^T\) and \((\cdot)^H\), respectively.

For general reviews of the problem, see [12, 16, 17, 20]. For further information on the NEVP see [1, 2, 10]. In some respects the present paper has grown out of [2] in which several design parameters are admitted and different scaling strategies for \( x \) and \( y \) are considered.

When applying continuation or iterative methods for the solution of (1) it is frequently the case that we have a solution at \( \rho = \rho_{-1} \equiv \rho_0 - h \) and seek the
solution at \( \rho = \rho_0 \) for some small positive step-size \( h \). Denoting partial derivatives with respect to \( \rho \) or \( \lambda \) by the corresponding subscripts, we shall therefore be interested in the partial derivatives \( \lambda_{\rho}, x_{\rho} \) and \( y_{\rho} \). We shall assume the existence of these partial derivatives as well as \( A_\lambda \) and \( A_{y_\rho} \). It will be assumed throughout that the eigenvalue \( A(\rho_0) \) is simple \([2]\) and that \( \det A(\lambda) \) is not identically zero. We denote the solution, \([\lambda(\rho_0), x(\rho_0), y(\rho_0)]\), to (1) at \( \rho_0 \) by \([\lambda, x, y]\) and sometimes use \( \lambda \) to emphasize the dependence on the eigenvalue parameters (as in \( A(\lambda) \) and \( g(\lambda) \)).

Some significant issues for the numerical methods considered here are:

1. The solution process should be numerically stable and efficient and, ideally, should produce the derivatives of the eigensolution with respect to \( \rho \) as byproducts.

2. More specifically, bounds for the condition numbers of the matrix operators in the solution processes should be available. These bounds indicate how some free variables in the solution process should be chosen.

3. For the sake of efficiency the solution \( \rho_0 = \rho_{-1} + h \) should be able to take advantage of information at \( \rho_{-1} \).

Except where specified otherwise, we use the 2-norm for vectors, the (induced) spectral norm for matrices and the spectral condition number \( \kappa(M) = \|M\|\|M^+\| \) (the ratio between the largest and smallest non-zero singular values of \( M \), with \((\cdot)^+\) denoting the Moore-Penrose generalized inverse \([8, p. 243]\)).

The methods described in this paper belong to a broad class of methods, reviewed in Section 2, which includes some classical methods. The basic idea is to construct a scalar valued function \( g \) whose zeros are the eigenvalues of \( A(\lambda) \) and to compute these zeros by Newton's method. In Sections 4–6 we introduce three new related methods, the BDS methods, which are suitable for numerical computation. In each of these methods the function \( g \) is computed using a rank-1 update of \( A \). The general theory of this approach, including a comparison with Rayleigh-quotient methods, is presented in Section 3. The three BDS methods (so named because they involve row/column Bordering, Deletion and Substitution respectively) correspond to different scaling schemes for the eigenvectors. As discussed in Section 7, they avoid the disadvantages associated with many other rank-1 update methods. Derivatives of the eigensolutions can be produced efficiently in these methods (see item (1) above) and bounds mentioned in item (2) are obtained. The use of information at \( \rho_{-1} \) (item (3) above) via updating methods and other details in the implementation of the methods are discussed briefly. The BDS methods are applied to some numerical examples in Section 8.

The main contribution of this paper consists of the complete and unified analysis of the BDS methods, in terms of the corresponding matrix equations,