An Extrapolation Method for the Efficient Composition of Maps with Applications to Non-Linear Oscillations

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Abstract — Zusammenfassung

An Extrapolation Method for the Efficient Composition of Maps with Applications to Non-Linear Oscillations. The determination of high order iterates of near identity maps is a problem of considerable interest in the theory of non-linear oscillations. Using tools from Asymptotic Analysis we derive an efficient extrapolation method and apply it in various situations, and in particular to the Lorenz equations.


1. Introduction

Consider a one-parameter family of maps \( P(x, \varepsilon) \), defined for \( x \in \mathbb{R}^d \), say, \( \varepsilon \) in a neighborhood of 0, with values in \( \mathbb{R}^d \), and satisfying \( P(x, 0) = x \). Let \( P^k \) denote the \( k \)-th iterate of \( P \). The goal of this paper is to provide an efficient numerical scheme for the computation of \( P^k \), for \( k \) large. In order to motivate the specific problem we treat below note the following. If \( k \) is arbitrarily large, but fixed, then \( P^k \to I \), as \( \varepsilon \to 0 \), where \( I \) denotes the identity map. However

\[
P^{1/\varepsilon} \longrightarrow I
\]

as \( \varepsilon \to 0 \), in general; here and in what follows \( \varepsilon \) is always restricted to those values, for which \( 1/\varepsilon \in \mathbb{N} \). Thus from the asymptotic point of view, \( P^{1/\varepsilon} \) is the simplest problem of interest.

The problem at hand is related to a class of problems in non-linear oscillations. Consider the system of differential equations

\[
\dot{x} = f(t, x, \varepsilon) = f^0(x) + \varepsilon \tilde{f}(t, x, \varepsilon)
\]  

(1),
together with the following assumptions: a) Every solution of Eq. (1)_0 is periodic of period 1, b) \( \tilde{f} \) is 1-periodic with respect to \( t \). There are a number of analytical tools available for Eq. (1). We mention the method of multiple scales, cf. Kevorkian (1981), and the method of averaging, cf. Bogoliubov (1965) or Kirchgraber (1978). The problem has been studied from a numerical point of view as well, cf. Taratynova (1960), Petzold (1981), Hoppensteadt (1981, 1983), Kirchgraber (1981, 1983).

In order to see the relation between Eq. (1) and our problem mentioned above, let \( \Phi_f(t, x) \) denote the solution of Eq. (1) with \( \Phi_f(0, x) = x \). Obviously \( P(x, \varepsilon) = \Phi_f(1, x) \) satisfies \( P(x, 0) = x \), and, since \( P^j(x, \varepsilon) = \Phi_f(j, x), \ j \in \mathbb{N} \), we conclude that the computation of \( P^{1/\varepsilon}(x, \varepsilon) \) in fact amounts to determine \( \Phi_f(1/\varepsilon, x) \). The more general problem, namely the efficient iteration of maps, is of interest even in view of some class of differential equations. If Eq. (1) is considered together with the assumptions a') all solutions of Eq. (1)_0 are periodic (but not necessarily with equal periods), b') \( f \) is autonomous, then we may introduce a section map (in the sense of Poincaré) and its iterates in order to study the behaviour of this system.

To begin our analysis we consider the nature of the dependence of \( P^{1/\varepsilon}(x, \varepsilon) \) on \( \varepsilon \). To provide some intuition first we mention that the simple but related function \( (1 + \varepsilon)^{1/\varepsilon} \) is analytic with respect to \( \varepsilon \) for \( \varepsilon \) in a neighborhood of \( \varepsilon = 0 \) and therefore admits a convergent expansion with respect to \( \varepsilon \). In the more general situation at hand a similar statement does not hold anymore in general, yet it will be proven that \( P^{1/\varepsilon}(x, \varepsilon) \) still admits at least an asymptotic expansion with respect to \( \varepsilon \) for \( \varepsilon \to 0 \), i.e. a representation of the type

\[
P^{1/\varepsilon}(x, \varepsilon) = \sum_{i=0}^{N-1} \varepsilon^i y^i(x) + O(\varepsilon^N)
\]

(cf. Eq. (17)). This fact calls for an extrapolation scheme for computing \( P^{1/\varepsilon}(x, \varepsilon) \). The traditional strategy would be to put \( \delta_i = a_i \cdot \varepsilon, i = 1, \ldots, N \), \( a_i > 1 \) and then to compute approximations of \( y^i(x) \) from the \( P^{1/\delta_i}(x, \delta_i) \). From an asymptotic point of view this approach is not satisfactory however, because the amount of work to compute \( P^{1/\delta_i}(x, \delta_i) \) is of the same order as for \( P^{1/\varepsilon}(x, \varepsilon) \). In order to get an asymptotically more efficient method we put \( \delta_i = a_i \cdot \sqrt[i]{\varepsilon}, j_i = 1/\delta_i \) for some natural \( k > 1 \) and introduce

\[
Q_i(x, \varepsilon) = P^{j_i}(x, \delta_i)
\]

(2)

\( l = 1, \ldots, N_1 \) with some suitable \( N_1 \). Note that asymptotically one needs much less iterations to compute the \( Q_i \)'s compared to the number of iterates involved in the computation of \( P^{1/\varepsilon}(x, \varepsilon) \). Fortunately, the knowledge of the \( Q_i \)'s for a reasonable choice of \( N_1 \) is sufficient to approximately compute \( P^{1/\varepsilon}(x, \varepsilon) \), as we shall see.

This procedure is related to the method proposed by Hoppensteadt and Miranker for differential equations, and has in fact been inspired by this paper. However, there is a major difference: Here we fully explore the asymptotic point of view. This is achieved by letting \( j_i \) depend on \( \varepsilon \). This way we are naturally led to higher-order methods and in addition the mathematical analysis is particularly nice.

This work was done while the second Author was visiting ETH.