Multivariate Rational Interpolation

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Abstract — Zusammenfassung

Multivariate Rational Interpolation. Many papers have already been published on the subject of multivariate polynomial interpolation and also on the subject of multivariate Padé approximation. But the problem of multivariate rational interpolation has only very recently been considered; we refer among others to [8] and [3].

The computation of a univariate rational interpolant can be done in various equivalent ways: one can calculate the explicit solution of the system of interpolatory conditions, or start a recursive algorithm, or calculate the convergent of a continued fraction.

In this paper we will generalize each of those methods from the univariate to the multivariate case. Although the generalization is simple, the equivalence of the computational methods is completely lost in the multivariate case. This was to be expected since various authors have already remarked [2, 7] that there is no link between multivariate Padé approximants calculated by matching the Taylor series and those obtained as convergents of a continued fraction.

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1. Algorithms for Univariate Rational Interpolation

Let the univariate function \( f(x) \) be given in the non-coincident interpolation points \( \{x_0, x_1, x_2, \ldots \} \). We consider the following problems:

\[
\begin{align*}
\text{calculate} & \quad \frac{P^n_j(x)}{Q^n_j(x)} = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{n} b_i x^i} \\
\text{such that} & \quad (f \cdot Q^n_j - P^n_j)(x_k) = 0 \quad \text{for} \quad k = j, \ldots, j + 2n
\end{align*}
\]
and

\[
\frac{p_{j+1,n}^n(x)}{Q_{j+1,n}^n(x)} = \frac{\sum_{i=0}^{n+1} a_i x^i}{\sum_{i=0}^{n} b_i x^i}
\]

such that \( (f \cdot Q_j^{n+1,n} - p_j^{n+1,n})(x_k) = 0 \) for \( k = j, \ldots, j + 2n + 1 \).

We shall say that the rational function "interpolates" the given function and by this we shall mean, also in the sequel of the text, that numerator and denominator of the rational function satisfy some linear conditions like (1) or (2).

This does not imply that the irreducible form of the rational function actually interpolates the given function at all the data, because, by constructing the irreducible form, a polynomial and hence some interpolation conditions may be cancelled in numerator and denominator of the rational interpolant.

The next theorem can be proved for the solutions of the problems (1) and (2). We denote \( f(x_k) \) by \( f_k \).

**Theorem 1.1:** The statements (a), (b), (c) and (d) are equivalent:

(a) \( \frac{p_{j,n}^n(x)}{Q_{j,n}^n(x)} \) and \( \frac{p_{j+1,n}^{n+1}(x)}{Q_{j+1,n}^{n+1}(x)} \) respectively satisfy (1) and (2)

\[
\frac{p_{j,n}^n(x)}{Q_{j,n}^n(x)} = \begin{vmatrix}
    f_j & x-x_j & f_j(x-x_j) & (x-x_j)^2 & f_j(x-x_j)^2 & \ldots & f_j(x-x_j)^n & f_j(x-x_j)^n \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
    f_{j+2n} & x-x_{j+2n} & & & & & & \\
    1 & x-x_j & f_j(x-x_j) & \ldots & (x-x_j)^n & f_j(x-x_j)^n \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \\
    1 & x-x_{j+2n} & & & & & & \\
\end{vmatrix}
\]

and

\[
\frac{p_{j+1,n}^{n+1}(x)}{Q_{j+1,n}^{n+1}(x)} = \begin{vmatrix}
    f_j & x-x_j & f_j(x-x_j) & \ldots & (x-x_j)^n & f_j(x-x_j)^n & (x-x_j)^{n+1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    f_{j+2n+1} & x-x_{j+2n+1} & & & & & & \\
    1 & x-x_j & f_j(x-x_j) & \ldots & (x-x_j)^{n+1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \\
    1 & x-x_{j+2n+1} & & & & & & \\
\end{vmatrix}
\]