A Bounding Approach to Calculating $\alpha^{1/p}$

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Abstract — Zusammenfassung

A Bounding Approach to Calculating $\alpha^{1/p}$. A simple method is exhibited for obtaining sequences which tend quadratically to $\alpha^{1/p}$ from above and below.

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Eine Einschließungsfolge für $\alpha^{1/p}$. Es wird eine einfache Methode angegeben, die quadratische von oben und von unten gegen $\alpha^{1/p}$ strebende Folgen erzeugt.

Let $x_r$ be an approximation to $\sqrt[p]{a}$, $a > 0$, where $p$ is an integer greater than 1. We can obtain a sequence $\{a_r\}$ of approximations to $\sqrt[p]{a}$ which tends quadratically to $\sqrt[p]{a}$ from above and a sequence $\{b_r\}$ of approximations which tends quadratically to $\sqrt[p]{a}$ from below. The basis of the method is the well-known result

\[
\text{Arithmetic mean} \geq \text{Geometric mean} \geq \text{Harmonic mean}
\]  

(1)

Thus, given $x_r$, a good approximation to $\sqrt[p]{a}$, we can take $a_{r+1}$ to be the arithmetic mean of the $p-1$ values $x_r$ ($>0$) and the value $(a/x_r^{p-1})$, i.e.,

\[
a_{r+1} = \frac{1}{p} \left[ (p-1)x_r + (a/x_r^{p-1}) \right], \quad r = 1, 2, \ldots
\]  

(2)

and $b_{r+1}$ to be the harmonic mean of the above quantities, i.e.,

\[
b_{r+1} = p \left[ \frac{p-1}{x_r} + \frac{x_r^{p-1}}{a} \right], \quad r = 1, 2, \ldots
\]  

(3)

and it follows from (1) that

\[
a_{r+1} \geq \sqrt[p]{a} \geq b_{r+1}.
\]

The formula (2) becomes the well-known Newton formula for the evaluation of $p$-th roots when $x_r = a_r$ and it is well-documented that, with exact arithmetic, the
sequence \( \{a_r\}, \ r > 1 \) is a monotonic decreasing sequence which tends to \( \sqrt[p]{a} \) from above. Similarly, if we put \( x_r = b_r \) in (3) then, with exact arithmetic, the sequence \( \{b_r\}, \ r > 1 \) is a monotonic increasing sequence which tends to \( \sqrt[p]{a} \) from below.

The object of this note is to point out the advantage of taking

\[
x_{r+1} = \frac{1}{2}(a_{r+1} + b_{r+1}).
\]

First we note that \( x_{r+1} \) is closer to \( \sqrt[p]{a} \) than either \( a_{r+1} \) or \( b_{r+1} \). For, if we assume \( x_r = \sqrt[p]{a}(1 + \epsilon) \) where \( \epsilon \) is small then

\[
a_{r+1} - \sqrt[p]{a} = \frac{1}{p} \left[ (p-1)(x_r - \sqrt[p]{a}) + \left( \frac{a}{x_r^{p-1}} \right) - \sqrt[p]{a} \right]
\]

\[
= \frac{1}{p} \left[ (p-1)\sqrt[p]{a} + \sqrt[p]{a} \left\{ 1 - (p-1)\epsilon + \frac{p(p-1)}{2} \epsilon^2 + \ldots - 1 \right\} \right]
\]

\[
= \sqrt[p]{a} \left[ \frac{1}{2} (p-1) \epsilon^2 + \ldots \right] = 0(\epsilon^2)
\]

Similarly

\[
b_{r+1} - \sqrt[p]{a} = \sqrt[p]{a} \left[ \frac{1}{2} (p-1) \epsilon^2 + \ldots \right] = 0(\epsilon^2)
\]

Now

\[
x_{r+1} - \sqrt[p]{a} = \frac{1}{2} (a_{r+1} + b_{r+1}) - \sqrt[p]{a}
\]

\[
= \frac{1}{3} \sqrt[p]{a} (p-1)(p-2) \epsilon^3 + \ldots = 0(\epsilon^3)
\]

which demonstrates that \( x_{r+1} \) is a better approximation to \( \sqrt[p]{a} \) than either \( a_{r+1} \) or \( b_{r+1} \). In particular, when \( p = 2 \), \( x_{r+1} - \sqrt[p]{a} = 0(\epsilon^4) \) so that, as far as square root evaluation is concerned, just a few more multiplications than in Newton’s method produces simultaneous upper and lower bounds for \( \sqrt[p]{a} \).

It is worth pointing out that the method is ideally suited for getting interval arithmetic estimates for \( p \)-th roots since all we are required to do is to get an upper interval estimate for \( a_r \) and a lower interval estimate for \( b_r \). Alternatively, the method can be used to provide an initial estimate for the interval Newton method.

In Tables 1 and 2 we give the calculations for \( \sqrt{2} \) and \( \sqrt[3]{2} \) respectively using a pocket calculator and starting in both cases with \( x_0 = 1 \).

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