A Note on the Stability of Rational Runge-Kutta Methods

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Abstract — Zusammenfassung

A Note on the Stability of Rational Runge-Kutta Methods. The present paper deals with the stability of rational Runge-Kutta methods in the numerical solution of stiff initial value problems. A natural stability requirement, called $A_0$-boundedness, is formulated. It is proved that there exist no rational Runge-Kutta methods which are either $A_0$-contractive or $A_0$-bounded.

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Key words: Stability, rational Runge-Kutta methods, numerical solution, stiff ordinary differential equations, $A_0$-boundedness, $A_0$-contractivity.


1. Introduction

1.1. Rational Runge-Kutta Methods

We shall deal with the numerical solution of the system of ordinary differential equations

$$\frac{dU(t)}{dt} = f(U(t)), \quad U(0) = u_0. \quad (1.1)$$

Here, $u_0 \in \mathbb{K}^n$, $f: \mathbb{K}^n \to \mathbb{K}^n$ are given and $U(t) \in \mathbb{K}^n$ is unknown (for $t > 0$). We are mainly interested in $\mathbb{K} = \mathbb{R}$, but it will be useful to consider $\mathbb{K} = \mathbb{C}$ as well. Throughout the paper $\langle \cdot, \cdot \rangle$ stands for an inner product on $\mathbb{C}^n$. The corresponding norm will be denoted by $| \cdot |$.

\[ u_{k+1} = u_k + \sum_{i=1}^{s} \sum_{j=1}^{s} w_{ij} \frac{g_i g_j}{\sum_{l=1}^{s} b_l g_l} \quad \left( \text{if } \sum_{l=1}^{s} b_l g_l \neq 0 \right), \]  
(1.2a)

\[ u_{k+1} = u_k \quad \left( \text{if } \sum_{l=1}^{s} b_l g_l = 0 \right), \]  

\[ g_i = h f \left( u_k + \sum_{j=1}^{i-1} a_{ij} g_j \right), \quad i = 1, 2, \ldots, s. \]  
(1.2b)

Here, \( h > 0 \) is the stepsize and \( u_k \approx U(kh) \) (\( k = 1, 2, \ldots \)). Further, \( a_{ij}, w_{ij}, b_l \) are real parameters and the expression \( \frac{a b}{d} \) is defined in the following way,

\[ \frac{a b}{d} = \frac{a \Re \langle b, d \rangle + b \Re \langle a, d \rangle - d \Re \langle a, b \rangle}{|d|^2}. \]  
(1.3)

In what follows we assume

\[ \sum_{i=1}^{s} b_i = 1, \]  
(1.4a)

\[ \sum_{i=1}^{s} \sum_{j=1}^{i} w_{ij} = 1, \]  
(1.4b)

\[ w_{ij} = 0 \quad (j > i), \]  
(1.5a)

\[ a_{ij} = 0 \quad (j \geq i). \]  
(1.5b)

(1.4a) is chosen in order to eliminate the ambiguity of the multiplication constant in the numerator and denominator of (1.2). Relation (1.4b) ensures that method (1.2) has an order of accuracy which is not less than one. Although \( w_{ij} (j > i) \) and \( a_{ij} (j \geq i) \) do not occur in (1.2) we shall assume (1.5a), (1.5b) throughout the paper for convenience.

We recall that implicit Runge-Kutta methods can possess very good stability and contractivity properties and in this respect they are suitable for stiff problems (cf. [3], [4]). But the main disadvantage of implicit Runge-Kutta methods is the large amount of computational effort (cf. [1], [2]). Rational Runge-Kutta methods seem to be worth considering due to the fact that they are explicit and do not even require the evaluation of the Jacobian of the function \( f \). Moreover, in [5], [10], it was shown that rational Runge-Kutta methods can have some stability properties that seem attractive in the numerical solution of stiff problems. But Sottas [8] and Spijker [9] pointed out that unfortunately rational Runge-Kutta methods have also some essential shortcomings with respect to stability. The question arises whether rational Runge-Kutta methods are good methods or not.

1.2. A Linear Test Problem

The standard stability analysis for linear methods is inappropriate for rational