A Strict Bound to the Condition Number of Bordered Positive Definite Matrices

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Abstract — Zusammenfassung

A Strict Bound to the Condition Number of Bordered Positive Definite Matrices. An easily computable strict bound is derived for the condition number in the $L_2$ norm of bordered positive definite matrices.

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Eine scharfe Schranke für die Konditionszahl in berandeten positiv-definiten Matrizen. Es wird eine leicht berechenbare scharfe Schranke für die $L_2$-Konditionszahl in berandeten positiv-definiten Matrizen hergeleitet.

1. Introduction

Bordered matrices appear in various areas of numerical analysis, including linear programming, regression analysis and methods for solving linear systems (for instance the classic escalator method or some formulations of the ABS class of direct methods recently proposed by Abaffy, Broyden and Spedicato [1]). Often the correction which defines the bordered matrix contains free parameters whose choice might be done in terms of some optimal conditioning criterion. In this paper we give a strict easily computable bound to the $L_2$ condition number (ratio of the largest to the smallest eigenvalue) of a (symmetric) positive definite matrix $D'$ obtained by bordering a (symmetric) positive definite matrix $D$:

$$D' = \begin{bmatrix} D & u \\ u^T & \alpha \end{bmatrix}$$

where $\alpha$ is a scalar, $u \in \mathbb{R}^i$, $D \in \mathbb{R}^{i \times i}$, $D' \in \mathbb{R}^{(i+1) \times (i+1)}$. A different proof of the given bound can be found in Spedicato [2]. The bound could also be obtained using the bounds for symmetric rank-two corrections to positive definite matrices established by Spedicato [3]. Applications of the bound to the ABS class have been considered by Spedicato [4], Bodon [5] and Deng [6]. In the following norms are $L_2$ norms.

2. Derivation of the Bound

The bound is derived as a consequence of the following Propositions.

**Proposition 1:** Let $A$ and $B$ be symmetric positive definite matrices of the same size. Then the following inequality is true:

$$\text{Cond}(B) \leq \text{Cond}(A) \text{Cond}(A^{-1/2} BA^{-1/2}).$$

(2.1)
Proof: Let $\gamma_1$ and $\gamma_2$ be the smallest and the largest eigenvalues of the (positive definite) matrix $A^{-1/2}BA^{-1/2}$. Then we have the two sided inequality

$$\gamma_1 I \leq A^{-1/2}BA^{-1/2} \leq \gamma_2 I.$$  \hfill (2.2)

Hence

$$\| B \| = \max (x^T B x: \| x \| = 1) = \max (y^T A^{-1/2} B A^{-1/2} y: \| A^{-1/2} y \| = 1) \leq \gamma_2 \max (y^T y: \| A^{-1/2} y \| = 1) = \gamma_2 \| A \|.$$  

Proceeding similarly we obtain $\| B^{-1} \| \leq \| A^{-1} \| /\gamma_1$ and (2.1) follows immediately.

**Proposition 2:** Let $D$ be a (symmetric) positive definite matrix and let $\mu_1, \mu_2$ be its smallest and largest eigenvalues. Let $\alpha$ be a positive scalar and define the matrix $\tilde{D}$ by

$$\tilde{D} = \begin{bmatrix} D & 0 \\ 0^T & \alpha \end{bmatrix}$$  \hfill (2.3)

then

$$\text{Cond} (\tilde{D}) = \max (\text{Cond} (D), \alpha /\mu_1, \mu_2 /\alpha).$$  \hfill (2.4)

Proof: Just observe that the eigenvalues of $\tilde{D}$ are $\alpha$ and the eigenvalues of $D$.

**Proposition 3:** Define the matrix $C$ by

$$C = \begin{bmatrix} I & v \\ v^T & 1 \end{bmatrix}$$  \hfill (2.5)

where $v$ is a vector such that $\| v \| < 1$. Then $C$ is positive definite and moreover

$$\text{Cond} (C) = (1 + \| v \|)/(1 - \| v \|).$$  \hfill (2.6)

Proof: Just observe that the eigenvalues of $C$ are $1 + \| v \|$, $1 - \| v \|$ and $1$ (with appropriate multiplicity).

We can now prove the main results.

**Theorem 1:** Let $D$ be a (symmetric) positive definite matrix and define $D'$ by (1.1). Then $D'$ is positive definite if and only if $\alpha$ satisfies

$$\alpha > u^T D^{-1} u.$$  \hfill (2.7)

Proof: The determinants of $D$ and $D'$ are related by relation

$$\det (D') = \det (D) (\alpha - u^T D^{-1} u).$$  \hfill (2.8)

Hence condition (2.7) is necessary for positive definiteness of $D'$. Now $D'$ can be written also in the form

$$D' = \tilde{D} + \tilde{e} \tilde{u}^T + \tilde{u} \tilde{e}^T$$  \hfill (2.9)

where $\tilde{e}$ is the unit vector in $\mathbb{R}^{i+1}$ and $\tilde{u} = (u^T, o)^T$. The correction to $\tilde{D}$ in (2.8) is symmetric rank-two of mixed type canonical form (see Brodlie, Gourlay and Greenstadt [7]). Thus it can be put in the form

$$\tilde{e} \tilde{u}^T + \tilde{u} \tilde{e}^T = a a^T - b b^T$$  \hfill (2.10)

for $a, b$ some vectors in $\mathbb{R}^{i+1}$. By the interlocking eigenvalue theorem it follows that