LIE SUPERALGEBRAS

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Results pertaining to the theory of representations of "classical" Lie superalgebras are collected in the survey.

PREFACE

A new area of mathematics — the theory of supermanifolds — arose in the 1970s. Its rapid growth was stimulated by fantastic prospects in physics: the possibilities of combining bosons and fermions into a single multiplet, of combining groups of inner and dynamical symmetries, and, finally, of combining all fundamental forces into a single field theory (see [13, 45, 47, 49, 50, 59, 62, 64, 90]). Moreover, in 1982 it was found that it was possible to formulate a model of field theory not containing singularities in the language of supersymmetries. An introduction to the theory of supermanifolds is presented in [5, 6, 32]. For an improved and corrected exposition and a list of some problems pertaining to this survey see [39, 45, 46]. The part of the theory of supermanifolds which now finds the greatest number of applications is the theory of Lie supergroups and superalgebras. Here we shall give a brief survey of results concerning the theory of representations of "classical" Lie superalgebras. For facts from linear algebra on superspaces see [32, 39]. We assume that the elements of the theory of representations of Lie algebras are known (see [12, 15, 21]).

The basic features of the theory of representations of simple Lie superalgebras make them kindred to Lie algebras in characteristic p, while if p = 2 there is almost no difference between algebras and superalgebras (see [113]). In particular, there is no complete reducibility and the Laplace—Casimir operators, which are of great help in describing representations of Lie algebras, play a modest role in the case of superalgebras [4, 82-86]. Methods from the theory of representations of infinite-dimensional Lie algebras of vector fields — the special vectors of Rudakov [53, 54] and analogues of Poincare's lemma — occupy center stage. By means of these methods it was possible, at least in principle, to determine how to solve the problem of O. Veblen on describing invariant differential operators acting on tensor fields on a manifold [22, 23], to refine it, to greatly generalize it, and in some cases to obtain a complete answer (see [9-11, 14, 26-29, 33-38, 65-72]).

0. Recollections

Regarding Algebras. As usual, we write **C**, **R**, **Z**, $\mathbf{Z}^+$, **N**, **H** and **O** for the complex numbers, real numbers, integers, nonnegative integers, positive integers, quaternions, and Cayley numbers, respectively. We denote by $|S|$ the power of a set S and by $\langle S\rangle$ the linear space generated by the set S. The base field is **C**.

Any finite-dimensional Lie algebra over C is the semidirect sum of a semisimple algebra and a maximal solvable ideal, while the semisimple algebra is the direct sum of simple algebras. The simple Lie algebras form the 3 classical series $\mathfrak{sl}$, $\mathfrak{o}$ and $\mathfrak{sp}$ and 5 exceptional Lie algebras.

All simple Lie algebras have the same structure. The Cartan subalgebra $\mathfrak{h}$ in a simple Lie algebra $\mathfrak{g}$ (i.e., the maximal nilpotent subalgebra coinciding with its normalizer) is commutative, and all Cartan subalgebras are conjugate relative to the action of the adjoint group. The Cartan subalgebra $\mathfrak{h}$ prescribes an $\mathfrak{h}^*$-gradation in $\mathfrak{g}=\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha$ and in finite-dimensional $\mathfrak{g}$-modules $M=\bigoplus_{\alpha\in P}M_\alpha$, whereby $\dim\mathfrak{g}_\alpha=\dim\mathfrak{g}_{-\alpha}$, where $\alpha\in\mathfrak{h}^*$. The elements of the sets $R, P\subset\mathfrak{h}^*$ are called the roots and weights, respectively.

The $\mathfrak{h}$-gradation in $\mathfrak{g}$ can be extended to a natural Z-gradation (in $\mathfrak{sl}(n)$ the degree of an

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element g is the number of the diagonal on which g lies over the main, zeroth diagonal), and it is possible to choose elements $X_i^{\pm}$, where $1 \leqslant i \leqslant \mathrm{rk}\,\mathfrak{g}$ of degree ±1, which generate $\mathfrak{g}$, whereby if $H_i = [X_i^+, X_i^-]$, then

$$[X_i^+, X_j^-] = \delta_{ij} H_i, \ [H_i, H_j] = 0,$$
$$(\mathrm{ad}\, X_i^{\pm})^{1-a_{ij}} (X_j^{\pm}) = 0. \qquad \text{(DR)}$$

The matrix (a_{ij}) is called the Cartan matrix and is conveniently assigned a Dynkin graph. The equations (DR) are the defining relations in $\mathfrak{g}$.

The weights are lexicographically ordered (relative to a fixed basis in $\mathfrak{h}$). In each finite-dimensional, irreducible module over a simple Lie algebra $\mathfrak{g}$ there is a 1-dimensional space (of leading vectors) corresponding to the highest (leading) weight.

The leading weight of a finite-dimensional module satisfies conditions that it be integral. For example, for $\mathfrak{sl}(n)$ the Cartan subalgebra consists of diagonal matrices; the index of the leading weight relative to the basis $\{e_{ii} - e_{i+1,i+1}\}$ must belong to $\mathbf{Z}^+$. The leading weight uniquely determines the irreducible modules; in particular, on the basis of χ it is possible to compute the character of the $\mathfrak{g}$-modules $L(\chi)$ with leading weight χ, i.e., the function $\mathrm{ch}\, M(\chi) = \sum_{\lambda \in P} \dim L(\chi)_\lambda e^\lambda$, where $L(\chi)_\lambda$ is the eigensubspace of the weight λ and $e^\lambda(h) = e^{\lambda(h)}$ for $h \in \mathfrak{h}$. Let $W(\mathfrak{g})$ be the Weyl group of the Lie algebra $\mathfrak{g}$, i.e., the group generated by reflections in hyperplanes of the space $\mathfrak{h}$ given by the roots. Then the following formula of H. Weyl holds:

$$\mathrm{ch}\, M(\chi) = \sum_{w \in W} \mathrm{sgn}\, w e^{w(\chi+\rho)} \Big/ \sum_{w \in W} \mathrm{sgn}\, w e^{w\rho} = \prod_{\alpha \in R^+} (1 + e^{-\alpha})^{\dim \mathfrak{g}_\alpha} \sum_{w \in W}{}' \mathrm{sgn}\, w e^{w\chi}, \text{ where } \rho(H_i) = 1.$$

It was found that finite-dimensional representations of simple Lie algebras are most simply described within the framework of the category $\mathcal{O}$ consisting of infinite-dimensional modules satisfying some natural conditions [15, 93]. An analogous category of modules can also be defined over Kac–Moody algebras. It is composed of nontrivial central extensions of Kac algebras consisting of the following two series of infinite-dimensional simple Lie algebras:

1) algebras of currents or loops $\mathfrak{g}^{(1)} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$ (the first name came from physicists, while the second is explained by the fact that $\mathfrak{g}^{(1)}$ = {mappings of the circle $S^1 \to \mathfrak{g}$ which can be expanded in a Fourier series where $\mathfrak{g}$ is a simple, finite-dimensional Lie algebra });

2) the Lie algebras $\mathfrak{g}_\varphi^{(m)} = \bigoplus_{k,j} \mathfrak{g}_j t^{mj+k}$ where $k = 0, \ldots, m-1$; $j \in \mathbf{Z}$, φ is an outer automorphism of order m of the simple, finite-dimensional Lie algebra $\mathfrak{g}$, and $\mathfrak{g}_j = \{g \in \mathfrak{g} \mid \varphi(g) = e^{2\pi i j/m} g\}$.

For a survey of Kac–Moody algebras see [110]. In particular, simple Kac algebras are given by the formula (DR) with an extended Cartan matrix (Dynkin graph), and for irreducible modules over them an analogue of H. Weyl's formula holds with the necessary alteration in the definition of W and restrictions on the leading weight.

In 1966 Kac distinguished an important class of Lie algebras related to the most different areas of mathematics and physics: simple Z-graded algebras of finite growth. (We recall that the growth or Gel'fand–Kirillov dimension of a Z-graded algebra A is $\lim_{n\to\infty}$((log $\dim \bigoplus_{|i|<n} I = \bigoplus_i I \cap A_i$, while simplicity of the algebra A means that there are no graded ideals).

Conjecture [107, 109, 110]. The simple Z-graded Lie algebras of finite growth over C are:

1) simple, finite-dimensional algebras;

2) Kac algebras;

3) Lie algebras of formal vector fields of types W, S, H, K;

4) the Witt algebra $W = \mathrm{Der}\, \mathbf{C}[t, t^{-1}]$.

The structure of the Lie algebras of these 4 classes is very different, which is naturally reflected in the theory of representations. The simplest representations — irreducible