Local Minimizer of a Nonconvex Quadratic Programming Problem

F. Mráz, České Budějovice

Received May 15, 1989; revised February 8, 1990

Abstract

Local Minimizer of a Nonconvex Quadratic Programming Problem. A modified Beale's algorithm is described which computes the local minimizer of any quadratic objective function subject to linear constraints. Some extensions are given, first of all the possibility of movement to the neighbouring local minimizer with a reduced objective function value in some special cases.

AMS Subject Classification: 90C20.

Key words: Quadratic programming problem, local minimizer.


1. Introduction

This paper deals with the minimization of a quadratic function of $n$ variables $x_j$ subject to linear constraints, i.e. with the problem

$$\min_{x} Q(x) = x^T C x + c^T x$$

on the set $M = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$,

where $C \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are given matrices and vectors.

Beale's original method [1] solves problem (1) with a convex objective function $Q$. If $Q$ is not convex, then algorithm [1] could produce a stationary point of $Q$ that is not even a local minimizer. Beale [2] improved his method in this respect. With two additional rules the algorithm must produce a local minimizer unless some linear term in the final expression for $Q$ happens to vanish. The next process in such a situation is described in [3]. Beale was not sure, however, whether it could cause cycling. Moreover, his discussion does not completely guarantee that the algorithm terminates at a local minimizer.

These two problems are investigated in Section 2 of the presented paper. A theoretical discussion is given of all possibilities for the values of the coefficients of any quadratic objective function subject to linear constraints.
In Section 3, a modified Beale's algorithm for finding a true local minimizer of the nonconvex problem is given. In Section 4, some extensions are indicated, first of all the possibility of movement to the next local minimizer with a reduced objective function value.

2. Beale's Method for a Nonconvex Objective Function

In accordance with Beale, assume we can express the basic variables in terms of the nonbasic variables

\[ x_k = d_{i0} + \sum_{j=1}^{n-m} d_{ij} z_j, \quad i = 1, 2, \ldots, m, \tag{2} \]

where \( z_j \) denotes \( x_{k_{m+j}} \).

On the assumption that \( d_{i0} \geq 0, i = 1, 2, \ldots, m \), we obtain an initial basic feasible solution

\[ x_{ki} = d_{i0}, \quad i = 1, 2, \ldots, m, \]
\[ x_{km+j} = z_j = 0, \quad j = 1, 2, \ldots, n - m. \tag{3} \]

We can use equations (2) to express the objective function in terms of the nonbasic variables

\[ Q(z) = Q(z_1, \ldots, z_{n-m}) = c_{00} + 2 \sum_{i=1}^{n-m} c_{io} z_i + \sum_{i=1}^{n-m} \sum_{j=1}^{n-m} c_{ij} z_i z_j. \tag{4} \]

Let us denote

\[ \text{Min} = \min \{ d_{i0} / |d_{ip}| : d_{ip} < 0 \}. \tag{5} \]

The theoretical background for the modified algorithm is presented in the following theorems.

**Theorem 1:** Let it hold for each \( i \in \{1, 2, \ldots, n-m\} \) that

\[ c_{io} > 0 \text{ or } (c_{io} = 0 \text{ and } (c_{ij} > 0 \text{ or } c_{j0} > 0, j = 1, \ldots, n-m)). \]

Then \( z = (0, 0, \ldots, 0) \) is a local minimizer of function \( Q \).

**Proof** follows immediately from expression (4).

**Theorem 2:** Let \( p \) be such that \( d_{ip} \geq 0 \) for each \( i \) and let it hold that \((c_{pq} = c_{pp} = 0 \text{ and } \exists q: c_{pq} < 0) \) or \((c_{pq} \leq 0, c_{pp} \leq 0, \text{ where at least one of inequalities holds sharply})\). Then the objective function is not bounded from below on set \( M \).

**Theorem 3:** Let there be \( p, q \) such that \( d_{q0} / |d_{qp}| = \text{Min} \) and \( c_{p0} \leq 0, c_{pp} \leq 0, \text{ where at least one of inequalities holds sharply. Then } Q(\hat{z}) < Q(0, \ldots, 0) \text{ holds, where } \hat{z} = (0, \ldots, 0, z_p = -d_{q0} / d_{qp}, 0, \ldots, 0) \text{ and } x_{kq} = 0. \)

**Theorem 4:** Let there be \( p \) such that \( c_{p0} < 0, c_{pp} > 0 \) and let \( d_{ip} \geq 0 \) for each \( i \). Then \( Q(\tilde{z}) < Q(0, \ldots, 0), \text{ where } \tilde{z} = (0, \ldots, 0, z_p = -c_{p0} / c_{pp}, 0, \ldots, 0) \text{ and } \frac{\partial Q(z)}{\partial z_p} = 0. \)

**Theorem 5:** Let there be \( p, q \) such that \( d_{q0} / |d_{qp}| = \text{Min} \) and let \( c_{p0} < 0 \) and \( c_{pp} > 0 \). Then we have: