On Multivariate Interpolation by Generalized Polynomials on Subsets of Grids

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Abstract — Zusammenfassung

On Multivariate Interpolation by Generalized Polynomials on Subsets of Grids. This note may be regarded as a complement to a paper of H. Werner [17] who has carried over Newton’s classical interpolation formula to Hermite interpolation by algebraic polynomials of several real variables on certain subsets of grids. Here generalized polynomials of several real or complex variables are treated. Recursive procedures are presented showing that interpolation by generalized multivariate polynomials is performed nearly as simply as interpolation by algebraic polynomials. Having in general the same approximation power, generalized polynomials may be better adapted to special situations. In particular, the results of this note can be used for constructing nonpolynomial finite elements since in that case the interpolation points usually are rather regular subsystems of grids. Though the frame is more general than in [17] some of our proofs are simpler. As an alternative method to evaluate multivariate generalized interpolation polynomials for rectangular grids a Neville-Aitken algorithm is presented.

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0. Notations

By \( \mathbb{N}_0 \) and \( \mathbb{N} \) we denote the set of nonnegative and positive integers, respectively. \( \mathbb{K} \) denotes either the field \( \mathbb{R} \) of real or the field \( \mathbb{C} \) of complex numbers. For the convex hull of a finite set \{\ldots\} we use the notation \( \text{con}\{\ldots\} \). Throughout this paper, void
suns are set equal to zero, void products and void determinants are set equal to one. CT and ECT mean complete and extended complete Chebyshev system, respectively.

1. Univariate Hermite Interpolation by Generalized Polynomials

It is well known that Newton’s classical interpolation polynomial has some advantages over other algorithmic approaches to the Hermite interpolation problem. This holds true for a generalization of Newton’s method ([1], [10], [11], [12], [13]) where generalized polynomials, i.e. linear combinations of functions forming an extended complete Chebyshev system are used. In this section we briefly repeat the univariate theory. In the next section most of its results will be carried over to multivariate interpolation on subsystems of grids owing a certain regularity property. We will treat only bivariate interpolation since the generalization to more than two variables has only notational difficulties. Mainly, we follow the lines of [17] using generalized instead of algebraic polynomials.

Let $G \subseteq \mathbb{K}$ be a region and let $(u_0, \ldots, u_m)$ be an extended complete $\mathbb{K}$-Chebyshev system (ECT-system) on $G$ of order $m$, i.e. $u_\mu \in C^m(G; \mathbb{K})$ and for $k = 0, \ldots, m$ and for all points $x_0, \ldots, x_m \in G$ the confluent generalized Vandermonde determinants do not vanish

$$V \begin{vmatrix} u_0, \ldots, u_k \\ x_0, \ldots, x_k \end{vmatrix} := \det \left( D^{x(x_\mu)} u_\mu(x_\mu) \right) \neq 0, \quad (1)$$

where $x_\mu :=$ multiplicity of $x_\mu$ in $(x_0, \ldots, x_{\mu-1})$ and $D := \frac{d}{dx}$.

Equivalently, $(u_0, \ldots, u_m)$ is an ECT-system on $G$ iff its generalized polynomials of order $k$ are solvent for any problem of Hermite interpolation ($k = 0, \ldots, m$):

given $(x_0, \ldots, x_k) \in G^{k+1}$ and a sufficiently smooth function $f: G \rightarrow \mathbb{K}$, compute a generalized polynomial

$$p_k f = \sum_{\mu=0}^{k} c_{\mu} \cdot u_\mu$$

that satisfies the interpolation conditions

$$D^{x(x_\mu)} p_k(x_\mu) = D^{x(x_\mu)} f(x_\mu) \quad (\mu = 0, \ldots, k).$$

It has been shown in [10], [11] and [13] that ($H_k$) is solved by

$$p_k f = p_{k-1} f + [x_0, \ldots, x_k] f \cdot r_{k-1} u_k = \sum_{\mu=0}^{k} [x_0, \ldots, x_\mu] f \cdot r_{\mu-1} u_\mu,$$

$$p_{-1} f := 0.$$

The coefficients are generalized divided differences which can be computed recursively.