A Study of B-Convergence of Linearly Implicit Runge-Kutta Methods

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Abstract — Zusammenfassung

A Study of B-Convergence of Linearly Implicit Runge-Kutta Methods. This paper deals with B-convergence analysis of linearly implicit Runge-Kutta methods as applied to stiff, semilinear problems of the form $y'(t) = Ty(t) + g(t, y)$. We analyse the discrepancy between the local and global order reduction. We show that linearly implicit Runge-Kutta methods of B-consistency order $q$ have the B-convergence order $q + 1$ for many singularly perturbed problems with constant stiff part. Numerical examples illustrate the theoretical results.

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Key words: Numerical analysis, ordinary differential equations, Runge-Kutta methods, stiff problems, B-convergence.

1. Introduction

1.1. Preliminaries

Consider the stiff initial value problem

$$y'(t) = f(t, y), \quad t \in [t_0, t_e], \quad f: [t_0, t_e] \times \mathbb{R}^n \to \mathbb{R}^n$$

$$y(t_0) = y_0$$

with $f$ satisfying a one-sided Lipschitz condition (with one-sided Lipschitz constant $v$)

$$\langle f(t, y_1) - f(t, y_2), y_1 - y_2 \rangle \leq v \| y_1 - y_2 \|^2,$$

$$(t, y_1), (t, y_2) \in [t_0, t_e] \times \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes an inner product with $\| x \| = \langle x, x \rangle^{1/2}$ as the corresponding norm.
For discretization a constant stepsize $h$ with $t_m = t_0 + m h, m = 0(1) N$, is chosen. As discretization methods the class of $s$-stage linearly implicit Runge-Kutta methods (see Strehmel/Weiner [10])

$$u_m^{(1)} = u_m$$

$$u_m^{(i)} = R_0^{(i)}(c_i h T) u_m + h \sum_{j=1}^{i-1} A_{ij}(h T) \left(f_j - T \cdot u_{m+1}^{(j)}\right), \quad i = 2, ..., s$$

$$u_{m+1} = R_0^{(s+1)}(h T) u_m + h \sum_{j=1}^{s} B_j(h T) \left(f_j - T \cdot u_{m+1}^{(j)}\right)$$

with $f_j = f(t_m + c_j h, u_{m+1}^{(j)})$ is considered. The vector $u_{m+1}$ approximates $y(t)$ at $t_{m+1} = t_0 + (m+1) h$. $R_0^{(i)}(z)$ is a rational approximation to $\exp (z)$ for $z \to 0$; $A_{ij}(z)$ and $B_j(z)$ are rational functions, $c_i$ are real parameters ($c_1 = 0, 0 < c_i \leq 1$ for $i = 2 (1) s$) and $T$ is an arbitrary $(n, n)$-matrix (usually an approximation to $f_y(t_m, u_m)$).

We symbolize method (1.1) by the tableau

\[
\begin{array}{cccc}
  c_2 & A_{21} \\
  c_3 & A_{31} & A_{32} \\
  \vdots & \vdots & \vdots & \cdots \\
  c_s & A_{s1} & A_{s2} & \cdots & A_{s,s-1} \\
  & B_1 & B_2 & \cdots & B_{s-1} & B_s
\end{array}
\]

Assume that the stability functions $R_0^{(i)}(z)$, $i = 2, ..., s + 1$, and the coefficients $A_{ij}(z)$ and $B_j(z)$ of the method fulfill the following conditions (see Strehmel/Weiner [10]):

(A1) The approximation order $r_i$ of $R_0^{(i)}(z)$ to $\exp (z)$ is sufficiently high ($r_i \geq p_i$: conventional stage order).

(A2) $R_0^{(i)}(z)$ has no pole in $\mathbb{C}^-$ and $|R_0^{(i)}(\infty)| < \infty, i = 2, ..., s + 1$.

(A3) $|A_{ij}(z)|$, $|z A_{ij}(z)|$, $|B_j(z)|$ and $|z B_j(z)|$ are uniformly bounded for $z \in \mathbb{C}^-$.

Let $v_{m+1}^{(i)}$ be the linearly implicit Runge-Kutta result from the transition $y(t_m) \to v_{m+1}^{(i)}$ and $l e_{m+1}^{(i)} = y(t_m + c_i h) - v_{m+1}^{(i)}$, $i = 2, ..., s$, the local error at the $i$-th stage and $l e_{m+1} = y(t_m + h) - v_{m+1}$ the full local error.

For the class $F$ of stiff semi-linear problems

$$y' = f(t, y) := T y + g(t, y)$$

where the constant $(n, n)$-matrix $T$ and the vector function $g: [t_0, t_e] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy

1. $\langle Tw, w \rangle \leq \mu [T] \|w\|^2, \mu [T] \leq 0$, for all $w \in \mathbb{R}^n$ (1.2a)

2. $\| g(t, y_1) - g(t, y_2) \| \leq L \| y_1 - y_2 \|, (t, y_1), (t, y_2) \in [t_0, t_e] \times \mathbb{R}^n$ (1.2b)

($\| g(t, y) \|$ may tend to infinity with increasing stiffness; see e.g. the model problem of Prothero and Robinson) the concept of $B$-consistency and $B$-convergence as introduced in Frank/Schneid/Ueberhuber [5] yields upper bounds for $l e_{m+1}^{(i)}$ and $l e_{m+1}$ in the form