A Special Method to Sample Some Probability Density Functions

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Abstract — Zusammenfassung

A Special Method to Sample Some Probability Density Functions. A special method is described to sample some probability density functions by help of their first derivatives. Two general theorems and some special applications, giving practical sampling procedures, are presented along with a number of illustrative examples.


Introduction

From the very early stage of development of Monte Carlo methods the question "how to sample a given probability density function" has always been central. During the last three decades a great number of general and special sampling techniques have been developed for such sampling [1—9], and with some exaggeration one could say that concerning this topic nothing further can be done. In fact there is not much hope of finding new general selecting methods, but surely there are special techniques appropriate for sampling one or another density functions, still not elaborated. In this paper such a special method is presented that uses the first derivative of the density function to be sampled. This method can be considered as an extension of a method of G. A. Mikhailov [10] since the latter is an interesting special case of our more general form.

Below two basic theorems giving the more general forms are stated along with some corollaries describing special cases. Finally some illustrative examples are given.

The Basic Theorems

Theorem 1: Let \( f(x) \) be a probability density function and assume that it is continuous if \( 0 \leq x < A \) (at \( A \) continuous from the left).
Let
\( a) \ f(x) = 0 \) if \( x \leq 0 \) or \( x > A \),
\( b) \ \frac{d f(x)}{d x} \equiv f'(x) \) be continuous if \( 0 \leq x < A \),
\( c) \ g(y) = \frac{1}{a} \left[ a f(y) + f'(y) \right] [1 - \exp (a(y - A))] \geq 0 \) if \( 0 \leq y < A \),

where \( a \) is an arbitrary real number (limited of course by the assumption itself).

If the conditions \( a) - c) \) hold then \( g(y) \) is again a probability density function (p.d.f.) and if \( \eta \) is a sample from \( g(y) \) then the random number
\[ \xi = -\frac{1}{a} \ln \left( e^{-ax} - (e^{-ax} - e^{-aA}) r \right) \]
is a sample from \( f(x) \). Here \( r \) is a random number distributed uniformly over \((0, 1)\).

In the following it will be referred to as \( r : u(0, 1) \).

Proof:
\[
\int_0^A g(y) \, dy = \int_0^A \frac{1}{a} \left[ a f(y) + f'(y) \right] [1 - \exp (a(y - A))] \, dy \\
= 1 + \frac{1}{a} f(A) - \frac{1}{a} \int_0^A \left[ f(y) e^y \right]' e^{-aA} \, dy = 1
\]
which shows that \( g(y) \) is really a probability density function. To get the density function \( P_\xi(x) \) of the random variable \( \xi \) one has to notice that
\[
P_\xi(x) = \int_0^A P_\xi(x \mid y) P_\eta(y) \, dy
\]
where
\[
P_\xi(x \mid y) = \begin{cases} \frac{e^{-ax} / (e^{-ax} - e^{-aA})}{y \leq x \leq A} & 0 \text{ elsewhere,} \end{cases}
\]
while
\[
P_\eta(y) = g(y).
\]
After substitution and integration one gets that
\[
P_\xi(x) = f(x),
\]
which is to be verified.

Theorem 2: Let \( f(x) \) be again a p.d.f., continuous for the positive \( x \) values and continuous from the right at zero.

Let
\( a) \ f(x) = 0 \) if \( x < 0 \),
\( b) \ f'(x) \) continuous if \( x > 0 \),
\( c) \ g(y) = \frac{1}{a} \left[ a f(y) + f'(y) \right] [1 - e^{ay}] \geq 0 \) if \( y \geq 0 \),
\( d) \ \lim_{y \to \infty} f(y) e^{ay} = 0 \),

where \( a \) is again an arbitrary real number.

If conditions \( a) - d) \) hold then \( g(y) \) is also a p.d.f. and if \( \eta \) is a sample from \( g(y) \) then
\[ \xi = -\frac{1}{a} \ln \left( 1 - (1 - e^{-a\eta}) r \right) \]