Rational Runge-Kutta Methods for Solving Systems of Ordinary Differential Equations

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Abstract — Zusammenfassung

Rational Runge-Kutta Methods for Solving Systems of Ordinary Differential Equations. Some nonlinear methods for solving single ordinary differential equations are generalized to solve systems of equations. To perform this, a new vector product, compatible with the Samelson inverse of a vector, is defined. Conditions for a given order are derived.


1. Introduction

Suppose the system of differential equations

\[ \frac{d\mathbf{y}}{dx} = \mathbf{g}(\mathbf{y}) \]  

is given, together with the initial values \( \mathbf{y}(x_0) = \mathbf{y}_0 \). Here

\[ \mathbf{y} = (y_1, y_2, \ldots, y_n)^T, \]

\[ \mathbf{g}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \ldots, f_n(\mathbf{y}))^T \]

and

\[ \mathbf{y}_0 = (y_{01}, y_{02}, \ldots, y_{0n})^T \]

are elements of \( \mathbb{R}^n \). Now we want to find an approximation \( \hat{\mathbf{y}} \) to the value of \( \mathbf{y} \) at the point \( x = x_0 + h \), by using a Runge-Kutta type method.

As was pointed out in a previous paper [13], the derivation of certain nonlinear Runge-Kutta methods can, for example, be done by using a nonlinear convergence accelerating rule such as the \( \varepsilon \)-algorithm. Unfortunately, these methods seldom extend to systems of equations [6, 9] and even if so, they are at most component applicable [12].
The nonlinear Runge-Kutta methods that will be treated in the present paper are formally defined by the equation

\[ \hat{y} = y_0 + h \left( \sum_{k=1}^{v} b_k \mathcal{g}^k \right), \]  

(1.2)

with

\[ \mathcal{g} = \mathcal{g} \left( y_0 + h \sum_{j=1}^{i-1} a_{ij} \mathcal{g}^j \right), \]  

(1.3)

whereas the linear methods are defined by

\[ \hat{y} = y_0 + h \sum_{k=1}^{v} c_k \mathcal{g}^k. \]  

(1.4)

Further on, it will be assumed that \( f_i(y) \) and all its derivatives exist for all \( i = 1, \ldots, n \), and that the Taylor expansions for \( y \) and \( \hat{y} \) are uniformly convergent, so that no summation difficulties will arise.

In the first section of this paper, a new definition of vector product and quotient will be given, in order to be able to extend nonlinear methods to systems of equations.

In the second section, this definition will be used to derive necessary conditions for methods of the type (1.2) to be of a certain order. This derivation is based on the approach of Butcher to the linear case [4], where the "elementary differentials" are defined and an operational method [2] is used to derive necessary conditions.

The third section will contain some remarks about the sufficiency of the conditions derived for the nonlinear Runge-Kutta methods. The last two sections are devoted to a discussion on stability, and some numerical results.

2. The Vector Product and Quotient

In the nonlinear formula (1.2), applied to a system of equations, quotients appear of the form

\[ \frac{a \cdot b}{d} \]  

(2.1)

where \( a, b, d \in \mathbb{R}^n \) and the dot stands for a vector multiplication to be defined.

This definition will be based on the Samelson inverse of a vector [3]:

\[ a^{-1} = \frac{a}{\| a \|^2}, \text{ where } \| a \|^2 = \sum_{i=1}^{n} a_i^2 = a^T a. \]

Let the vector product \( a \cdot b \) be defined as the \( n \) by \( n \) matrix

\[ a \cdot b = a b^T + b a^T - a^T b I_n = b \cdot a. \]  

(2.2)