The Approximate Dirichlet Domain Decomposition Method. Part II: Applications to 2nd-order Elliptic B.V.P.s

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Abstract — Zusammenfassung

The Approximate Dirichlet Domain Decomposition Method. Part II: Applications to 2nd-order Elliptic B.V.P.s. In the first part of this article series, we had derived Domain Decomposition (DD) preconditioners containing three block matrices which must be specified for specific applications. In the present paper, we consider finite element equations arising from the DD discretization of plane, symmetric, 2nd-order, elliptic b.v.p.s and specify the matrices involved in the preconditioner via multigrid and hierarchical techniques. The resulting DD-PCCG methods are asymptotically almost optimal with respect to the operation count and well suited for parallel computations on MIMD computers with local memory and message passing. The numerical experiments performed on a transputer hypercube confirm the efficiency of the DD preconditioners proposed.

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1. Introduction

Recently domain decomposition (DD) methods have received much attention in connection with rising interest in parallel computation (see the proceedings of DD conferences [5, 6, 18]). However, some basic ideas go back to the paper [19] of H. A. Schwarz (1869), or, at least, to substructuring techniques developed by the engineers in the finite element analysis of complicated solid structures at the beginning of the sixties (see, e.g., [18]). Later on these ideas have been used to derive preconditioners for the conjugate gradient (CG) method. At present the studies are concentrated to such preconditioners which take advantage of one or another
architecture of parallel computers (see the references given in the first part of this paper [15]).

Let us first summarize the results of the first part [15] of this paper in which we had presented a purely algebraic approach to the construction of Dirichlet DD preconditioners. In the usual nodal basis \( \Phi = (\varphi_1, \ldots, \varphi_{N_c}, \varphi_{N_c+1}, \ldots, \varphi_{N=N_c+N_t}) \), the finite element equations \( K_u = f \) can be rewritten in the block form

\[
\begin{pmatrix}
K_c & K_{cI} \\
K_{IC} & K_I
\end{pmatrix}
\begin{pmatrix}
u_c \\
u_I
\end{pmatrix} =
\begin{pmatrix}
f_c \\
f_I
\end{pmatrix},
\]

(1.1)

where the indices "I" and "C" correspond to the nodes belonging to the interior \( \bigcup_{i=1}^{p} \Omega_i \) of the subdomains into which the domain \( \Omega \) is decomposed (\( \Omega_i \cap \Omega_j = \emptyset \) for \( i \neq j \)) and to the coupling boundaries \( \Gamma_c = \bigcup \partial \Omega_i \setminus \Gamma_D \) (\( \Gamma_D \) is the Dirichlet boundary), respectively. The stiffness matrix \( K \) is supposed to be symmetric and positive definite throughout the paper. In Section 3 of [15], we derive the finite element equations in the exact and the approximate discrete harmonic basis \( \Phi \hat{\nu} \) and \( \Phi \tilde{\nu} \) defined by the basis transformation matrices

\[
\hat{\nu} = (V_c \hat{\nu}_I) = \begin{pmatrix} I_c & 0 \\ -K_I^{-1}K_{IC} & I_I \end{pmatrix} \quad \text{and} \quad \tilde{\nu} = (\tilde{V}_c \tilde{\nu}_I) = \begin{pmatrix} I_c & 0 \\ -C_I^{-1}K_{IC} & I_I \end{pmatrix},
\]

respectively, where \( C_I \) is some non-singular matrix of the dimension \( N_I \times N_I \) called the “basis transformation matrix" in the following. These two basis transformations result in the two representations

\[
\nu = \hat{\nu}_c \oplus \hat{\nu}_I \quad \text{and} \quad \nu = \tilde{\nu}_c \oplus \tilde{\nu}_I
\]

of the finite element space \( \nu = \text{span}(\Phi) \) as orthogonal (with respect to the energy inner product) sum of the subspaces \( \hat{\nu}_c = \text{span}(\Phi \hat{\nu}_c) \) and \( \hat{\nu}_I = \text{span}(\Phi \hat{\nu}_I) \) and as direct sum of the subspaces \( \tilde{\nu}_c = \text{span}(\Phi \tilde{\nu}_c) \) and \( \tilde{\nu}_I = \text{span}(\Phi \tilde{\nu}_I) \), respectively. In section 4 of [15], we show that

\[
\cos^2 \vartheta = \langle \hat{\nu}_c, \hat{\nu}_I \rangle = \mu/(1 + \mu),
\]

where \( \mu = \rho(S_c^{-1}T_c) \) is the spectral radius of \( S_c^{-1}T_c \), \( S_c = K_c - K_{CI}K_I^{-1}K_{IC} \) is the Schur complement, and \( T_c \) denotes the operator \( K_{CI}(C_I^{-T} - K_I^{-1})K_I(C_I^{-1} - K_I^{-1})K_{IC} \). This observation enables us to derive the DD preconditioner

\[
B = \begin{pmatrix} I_c & K_{CI}C_I^{-T} \\ 0 & I_I \end{pmatrix} \begin{pmatrix} B_c & 0 \\ 0 & B_I \end{pmatrix} \begin{pmatrix} I_c & 0 \\ 0 & I_I \end{pmatrix}
\]

(1.2)

and to prove the estimate

\[
\gamma_1 \gamma_I \leq \rho(B^{-1}K) \leq \gamma_1 \gamma_I (\sqrt{\mu} + \sqrt{1 + \mu})^2
\]

(1.3)

of the relative spectral condition number \( \rho(B^{-1}K) \) from above and below, where \( \gamma_1 = \min \{ \gamma_I, \gamma_C \} \) and \( \gamma_I = \max \{ \gamma_I, \gamma_C \} \). The block preconditioners \( B_c \) and \( B_I \) satisfy the spectral equivalence inequalities

\[
\gamma_I B_I \leq K_I \leq \tilde{\gamma}_I B_I \quad \text{and} \quad \gamma_C B_c \leq S_c + T_c \leq \tilde{\gamma}_C B_c,
\]

(1.4)

with positive spectral equivalence constants \( \gamma_C, \gamma_I, \tilde{\gamma}_C, \tilde{\gamma}_I \) (see Section 5 in [15]).