STRONG MINIMALITY OF ABNORMAL GEODESICS FOR 2-DISTRIBUTIONS

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Abstract. We investigate the local length minimality (by the $W_{1,1}$- or $H_1$-topology) of abnormal sub-Riemannian geodesics for rank 2 distributions. In particular, we demonstrate that this kind of local minimality is equivalent to the rigidity for generic abnormal geodesics, and introduce an appropriate Jacobi equation in order to compute conjugate points. As a corollary, we obtain a recent result of Sussmann and Liu about the global length minimality of short pieces of the abnormal geodesics.

1. Introduction

In this paper we study abnormal sub-Riemannian geodesics. Let us recall that a sub-Riemannian structure on a Riemannian manifold $M$ is defined by a bracket-generating distribution $\mathcal{D}$ on $M$, or one possessing full Lie rank. A locally Lipschitzian path $q(\tau)$ ($\tau \in [0,T]$) is admissible if its tangents lie in $\mathcal{D}$ for almost all $\tau \in [0,T]$. Given two points $q^0$ and $q^1$, one can set out the problem of finding the minimal (i.e., length-minimizing) admissible path connecting $q^0$ with $q^1$.

An essential distinction of this setting from the classical Riemannian case is that the space of all locally Lipschitzian paths connecting $q^0$ with $q^1$ has the structure of a Banach manifold with minimal paths that are critical points of the length functional, or Riemannian geodesics on the manifold $M$, whereas the space of admissible paths is not, in general, a manifold and may have singularities. These singularities correspond to the so-called abnormal geodesics. In fact, these abnormal geodesics do not depend on the Riemannian structure and are determined by the distribution $\mathcal{D}$.

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The term "abnormal" comes from the calculus of variations since the problem of finding minimal admissible paths can be reformulated as the Lagrange problem of the calculus of variations. The Euler-Lagrange equation for the Lagrange problem is called a geodesic equation; its solutions are extremals of the Lagrange problem or sub-Riemannian geodesics. In particular, abnormal extremals with a vanishing Lagrange multiplier for the (length) functional are abnormal geodesics.

For a long time abnormal sub-Riemannian geodesics were not treated by geometers as proper candidates for minimizers until Montgomery gave in [15] an example of a minimal admissible path which does not correspond to any normal sub-Riemannian geodesic. Later another example was constructed by Kupka [13]), and Sussmann established in [19] the minimality of short abnormal geodesic subarcs for generic 2-distributions in $\mathbb{R}^4$. Later Sussmann and Liu generalized the last result to the 2-distributions in $\mathbb{R}^n$ [20].

Another approach to the investigation of weak (i.e., $W_{1,\infty}$-local) 1 minimality of abnormal extremals of the Lagrange problem and the abnormal sub-Riemannian geodesics was suggested by the authors in [5], [6]. It is a kind of Legendre-Jacobi-Morse-type theory of a second variation for abnormal extremals of the Lagrange problem and the abnormal geodesics and, therefore, deals with geodesics of an arbitrary length. Among the results established in [6] are second-order Jacobi-type conditions of weak minimality for abnormal geodesics, which turned out to be also conditions of rigidity for the corresponding abnormal geodesic paths. Recall that rigidity means that an admissible path is isolated (up to a reparametrization) in $W_{1,\infty}$-topology in the set of all admissible paths connecting the given points $q^0, q^1 \in M$; see [7]. As was demonstrated in [6], the rigidity conditions follow from a general necessary/sufficient condition for critical points of a smooth mapping to be isolated at the corresponding critical level. Developing in [6] the Jacobi-Morse-type approach to abnormal geodesics, the authors introduced the notions of Morse index and nullity and derived explicit formulas for these invariants. This made it possible to establish local rigidity of an abnormal geodesic meeting the Strong Generalized Legendre Condition.

In this paper we are going to establish sufficient conditions for $W_{1,1}$-local minimalitly of abnormal sub-Riemannian geodesic paths. We call it strong minimality, although it differs from the traditional definition of strong minimality in the calculus of variations, which is a $C^0$-local

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1By $W_{1,k}[0,T]$, $1 \leq k \leq \infty$, we denote the spaces of absolutely continuous (vector-) functions on $[0,T]$ ($T < \infty$) whose derivatives belong to $L^k[0,T]$. They become Banach spaces when provided with the norms $\|w(\cdot)\|_{1,k} = (\|w(0)\| + \|\dot{w}(\cdot)\|_{L^k})^{1/2}$. In particular, $W_{1,1}[0,T]$ is the space of absolutely continuous functions, and $W_{1,2}$ is Sobolev space $H^1[0,T]$. 