CRITERIA OF WEIGHTED INEQUALITIES IN ORLICZ
CLASSES FOR MAXIMAL FUNCTIONS DEFINED ON
HOMOGENEOUS TYPE SPACES

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ABSTRACT. The necessary and sufficient conditions are derived in or-
der that a strong type weighted inequality be fulfilled in Orlicz classes
for scalar and vector-valued maximal functions defined on homoge-
neous type spaces. A weak type problem with weights is solved for
vector-valued maximal functions.

§ 0. INTRODUCTION

The main goal of this paper is to obtain criteria for the validity of an
inequality of the form
\[ \int_X \varphi(Mf(x))w(x) \, d\mu \leq c \int_X \varphi(f(x))w(x) \, d\mu \]  
(0.1)
for maximal functions defined on homogeneous type spaces.

The solution of a strong type one-weighted problem for classical maximal
functions in reflexive Orlicz spaces was obtained for the first time by R.
Kerman and A. Torchinsky [5]. This investigation was further developed in
[6], [7]. Quite a simple criterion established in this paper in the general case
is the new one for Hardy–Littlewood–Wiener maximal functions as well.
Our present investigation is a natural continuation of the nonweighted case
[1], [2], [3], [4]. Conceptually it is close to [2], [8], [9], [15], [16].

For vector-valued Hardy–Littlewood–Wiener maximal functions in the
nonweighted case the boundedness in \( L^p \), \( 1 < p < \infty \), was established
in [9]. A weighted analogue of this result was obtained in [10] (see also
[11], [12], [13]). Finally, we should mention [14], [15], [16] containing the
full descriptions of functions \( \varphi \) and a set of weight functions ensuring the
validity of a weak type weighted inequality for maximal functions.

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We shall now make some comments on how this paper is organized. The introduction contains some commonly known facts on homogeneous type spaces and weight functions defined in such spaces. Here the reader will also find the definition of quasi-convex functions and a brief discussion of some of their simple properties. The main results are formulated at the end of the introduction. In §1 we describe the class of quasi-convex functions, and functions which are quasi-convex to some degree less than 1. A number of useful properties to be used in our further discussion are established for such functions. Further sections contain the proofs of the main results.

Let \((X, d, \mu)\) be a homogeneous type space (see, for example, [17], [19]). It is a metric space with a complete measure \(\mu\) such that the class of compactly supported continuous functions is dense in the space \(L^1(X, \mu)\). It is also assumed that there is a nonnegative real-valued function \(d : X \to \mathbb{R}^+\) satisfying the following conditions:

(i) \(d(x, x) = 0\) for all \(x \in X\);
(ii) \(d(x, y) > 0\) for all \(x \neq y\) in \(X\);
(iii) there is a constant \(a_0\) such that \(d(x, y) \leq a_0 d(y, x)\) for all \(x, y\) in \(X\);
(iv) there is a constant \(a_1\) such that \(d(x, y) \leq a_1 (d(x, z) + d(z, y))\) for all \(x, y, z\) in \(X\);
(v) for each neighborhood \(V\) of \(x\) in \(X\) there is an \(r > 0\) such that the ball \(B(x, r) = \{y \in X; d(x, y) < r\}\) is contained in \(V\);
(vi) the balls \(B(x, r)\) are measurable for all \(x\) and \(r > 0\);
(vii) there is a constant \(b\) such that \(\mu B(x, 2r) \leq b \mu B(x, r)\) for all \(x \in X\) and \(r > 0\).

An almost everywhere positive locally \(\mu\)-summable function \(w : X \to \mathbb{R}^+\) will be called a weight function. For an arbitrary \(\mu\)-measurable set \(E\) we shall assume

\[
we = \int_E w(x) \, d\mu.
\]

By definition, the weight function \(w \in A_p(X) (1 \leq p < \infty)\) if

\[
\sup_B \left( \frac{1}{\mu B} \int_B w(x) \, d\mu \right) \left( \frac{1}{\mu B} \int_B (w(x))^{-1/p-1} \, d\mu \right)^{p-1} < \infty \quad \text{for} \quad 1 < p < \infty,
\]

where the supremum is taken over all balls \(B \subset X\) and

\[
\frac{1}{\mu B} \int_B w(x) \, d\mu \leq \text{ess inf}_{y \in B} w(y) \quad \text{for} \quad p = 1.
\]

In the latter inequality \(c\) does not depend on \(B\). The above conditions are analogues of the well-known Muckenhoupt's conditions.