CONTINUOUS TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS WITH DELAYS

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ABSTRACT. The aim of this paper is to find the class of continuous pointwise transformations (as general as possible) in the framework of which Kummer's transformation $z(t) = g(t)y(h(t))$ represents the most general pointwise transformation converting every linear homogeneous differential equation of the $n$th order into an equation of the same type. Further, some forms of these equations having certain subspaces of solutions are considered.

Let $I = [a, b)$, $J = [c, d)$ be intervals, where $b, d$ may be infinite. Further, let

$$t = f_1(x, y), \quad z = f_2(x, y)$$

where $(x, y) \in I \times \mathbb{R}$ and denote by $F = (f_1, f_2)$ a pointwise transformation of $I \times \mathbb{R}$ into $U \subset \mathbb{R}^2$.

In this article we shall study transformations $F$ of a linear homogeneous differential equation of the $n$th order with $m$ delays

$$y^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x)y^{(i)}(x) + \sum_{i=0}^{n-1} \sum_{j=1}^{m} q_{ij}(x)y^{(i)}(\tau_j(x)) = 0 \quad (1)$$
on $[x_0, b)$, where the initial set $E_{x_0} = [a, x_0]$, $p_i$, $q_{ij}$, $\tau_j \in C([x_0, b])$, $\tau_j(x) < x$ on $[x_0, b)$, and $q_{ij} \neq 0$ on $[x_0, b)$ for a pair $(i, j)$ ($i = 0, \ldots, n; j = 1, \ldots, m$). We wish to derive the form of such a transformation $F$ which converts (in the sense of a pointwise transformation of solutions) every equation (1) into an equation of the same type, i.e.,

$$z^{(n)}(t) + \sum_{i=0}^{n-1} r_i(t)z^{(i)}(t) + \sum_{i=0}^{n-1} \sum_{j=1}^{m} s_{ij}(t)z^{(i)}(\mu_j(t)) = 0 \quad (2)$$

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on \([t_0,d]\), where \(E_{t_0} = [c,t_0], r_i, s_{ij}, \mu_j \in C^0([t_0,d]), \mu_j(t) < t\) on \([t_0,d]\), and \(s_{ij} \neq 0\) on \([t_0,d]\) for a pair \((i,j)\) \((i = 0, \ldots, n, j = 1, \ldots, m)\). Notice that solutions \(y(x)\) (resp. \(z(t)\)) of equation (1) (resp. (2)) are functions defined on the whole \(I\) (resp. \(J\)) and the space of solutions of both equations has an infinite dimension.

Next, let \(W(y_1, \ldots, y_k)(x)\), where \(y_1, \ldots, y_k \in C^{k-1}(I)\), be the Wronski determinant of functions \(y_1, \ldots, y_k\) at \(x \in I\).

We denote the following hypotheses concerning \(F\):

\(\text{(H1)}\) \(F\) is a \(C^n\)-diffeomorphism of \(I \times \mathbb{R}\) onto \(J \times \mathbb{R}\);

\(\text{(H_1')}\) \(F\) is a homeomorphism of \(I \times \mathbb{R}\) onto \(J \times \mathbb{R}\);

\(\text{(H_2)}\) for every equation (1) there exists an equation (2) such that \(F\) converts, pointwise,

\((i)\) every nontrivial solution \(y(x)\) of (1) into a nontrivial solution \(z(t)\) of (2);

\((i')\) every nontrivial solution \(y(x)\) of (1) into a \(C^n\) function \(z(t)\) defined on \(J\);

\((ii)\) every \(k\)-tuple \(y_1, \ldots, y_k\) of solutions of (1) satisfying \(W(y_1, \ldots, y_k)(x) \neq 0\) on \(I\) into a \(k\)-tuple \(z_1, \ldots, z_k\) of solutions of (2) satisfying \(W(z_1, \ldots, z_k)(t) \neq 0\) on \(J\), where \(k \in \{2, \ldots, n+1\}\) is a suitable number;

\((iii)\) every function \(y \circ \tau_j\), where \(y\) is a solution of (1), into a function \(z \circ \mu_j\), where \(z\) is a solution of (2) \((j = 1, \ldots, m)\).

Assuming \(q_{ij} \equiv 0\) on \([x_0,b)\) for each pair \((i,j)\) in (1) we obtain a differential equation without any delay. The problem of the most general pointwise transformation converting any such equation (1) into an equation (2) (with \(s_{ij} \equiv 0\) on \([t_0,d]\)) was first solved by P. Stäckel in [1]. He proved that under hypotheses \((\text{H_1})\) and \((\text{H_2})(\text{i})\), \(F\) has the form

\[ t = f(x), \quad z = g(t)y \quad \text{for } n \geq 2 \quad t = f(x), \quad z = g(t)y^\lambda, \quad \lambda > 0 \quad \text{for } n = 1 \]

where \(f\) is a \(C^n\)-diffeomorphism of \(I\) onto \(J\), \(g \in C^n(J), g(t) \neq 0\) on \(J\). Recently M. Čadek has shown (see [2]) that the assumption of differentiability of \(F\) is not necessary and the form of \(F\) remains preserved also under \((\text{H_1}), (\text{H_2})(\text{i'}), \) and \((\text{H_2})(\text{ii}), \) where \(k = n\) (for more details see [3]).

Provided \(q_{ij} \neq 0\) on \([x_0,b)\) for a pair \((i,j)\) V. Tryhuk proved in [4] that assuming \((\text{H_2})(\text{i}), (\text{H_2})(\text{ii}), \) and \((\text{H_2})(\text{iii}), \) \(F\) has the form \(t = f(x), \quad z = g(t)y\) with \(g\) and \(f\) having the same properties as above and, moreover, \(f'(x) > 0\) on \(I\) and \(f \circ \tau_j = \mu_j \circ f\) on \(I\) for \(j = 1, \ldots, m\). The aim of this paper is to weaken the assumption of differentiability of \(F\) as M. Čadek did for equations without delays.

**Proposition 1.** Let hypotheses \((\text{H_1'})\) and \((\text{H_2})(\text{i'})\) be fulfilled. Then \(f_1\) is a homeomorphism between \(I\) and \(J\) not depending on \(y\).