ON A RELATIONSHIP BETWEEN THE INTEGRABILITIES OF VARIOUS MAXIMAL FUNCTIONS

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Abstract. It is shown that the right-sided, left-sided, and symmetric maximal functions of any measurable function can be integrable only simultaneously. The analogous statement is proved for the ergodic maximal functions.

Introduction. We deal with integrable functions on $T = [0, 2\pi)$ and assume that they are extended to $2\pi$-periodic functions on the whole line $\mathbb{R}$. The class of such functions will be denoted by $L$. One can also consider the functions of $L$ to be defined on the unit circle in the complex plane.

If a measurable set $E \subset \mathbb{R}$ is such that $f$ is a $2\pi$-periodic function and $f \in L$, then we assume that

$$
\int_E f \, d\nu = \int_{E \cap T} f \, d\nu
$$

($\nu$ denotes the Lebesgue measure on the line).

We shall say that a subset $\Delta \subset \mathbb{R}$ is a segment of $T$ if it is the preimage of an open arc of the unit circle by the exponential function. The set of such segments is denoted by $\mathcal{E}$. If $\Delta \in \mathcal{E}$, $\Delta \neq \mathbb{R}$ and $(a, b)$ is a connected component of $\Delta$, then we shall write $\Delta = (a, b)$, which should not cause any confusion. Obviously, in that case $|\Delta| = b - a$.

Let $x \in T$. We introduce the following notations of subsets of $\mathcal{E}$:

- $\mathcal{E}_0(x) = \{(a, b) \in \mathcal{E} : a < x < b\}$,
- $\mathcal{E}_1(x) = \{(a, b) \in \mathcal{E} : b = x\}$,
- $\mathcal{E}_2(x) = \{(a, b) \in \mathcal{E} : a = x\}$,
- $\mathcal{E}_3(x) = \{(a, b) \in \mathcal{E} : \frac{a + b}{2} = x\}$.

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Consider the maximal operators $M_j$, $j = 0, 1, 2, 3$, defined by the equalities

$$M_j(f)(x) = \sup_{\Delta \in \mathcal{F}_j(x)} \frac{1}{|\Delta|} \left| \int_{\Delta} f dv \right|, \quad f \in L.$$ 

It is well known that $f \in L \Rightarrow M_j(f) \in L$, $j = 0, 1, 2, 3$, and if $f \geq 0$, then the inverse implication is true (see [1], [2]). But, in general, one cannot write explicitly the set of functions $f$ for which $M_j(f)$ is integrable (in connection with this see [2], [3]). In this paper we shall show that for an arbitrary $f \in L$ the functions $M_j(f)$, $j = 0, 1, 2, 3$, can be integrable only simultaneously. An analogous statement is proved for the ergodic maximal functions in §2.

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§ 1. Obviously, $M_0(f) \geq M_j(f)$, $j = 1, 2, 3$. We shall prove the following theorems.

**Theorem 1.** Let $f \in L$ and $M_1(f) \notin L$. Then $M_0(f) \notin L$.

**Theorem 2.** Let $f \in L$. Then

$$M_1(f) \notin L \Leftrightarrow M_2(f) \notin L.$$ 

Since $M_0(f) \leq M_1(f) + M_2(f)$, Theorems 1 and 2 enable us to conclude that the functions $M_j(f)$, $j = 1, 2, 3$, are nonintegrable whenever $M_0(f)$ is nonintegrable.

We begin by proving some lemmas. Their proofs are given in the form simplifying their extension to the ergodic case.

Let $M$ be the operator

$$M(f)(x) = \sup_{a < x} \frac{1}{x - a} \int_a^x f dv, \quad f \in L.$$ 

Evidently, $\{x \in \mathbb{R} : M(f)(x) > t\} = (M(f) > t)$ is an open subset of $\mathbb{R}$ for each $t$.

**Lemma 1.** Let $f \in L$, $t > 0$, and let $(a, b)$ be a finite (i.e., $a \neq -\infty$, $b \neq \infty$) connected component of $(M(f) > t)$. Then we have

$$\frac{1}{x - a} \int_a^x f dv > t$$

for each $x \in (a, b)$.

This lemma was actually proved in [4] but we give it here for the sake of completeness.