ROOTS OF THE PHASE OPERATORS

G. JORJADZE AND I. SARISHVILI

ABSTRACT. The kth root taken of the bosonic phase operator is considered. This leads to the extension of the Hilbert space. In the case $k = 2$ an ordinary fermionic extension arises. Particles whose statistics depends on $k$ are introduced for other values of $k$.

1. PHASE OPERATORS

The problem of polar decomposition of the creation and annihilation operators $(a^*, a)$ of a harmonic oscillator goes back to Dirac [1]. In this section some aspects of this problem will be reviewed [2].

Let $h$ be the oscillator hamiltonian

$$h = \frac{p^2 + q^2}{2}$$

(1.1)

on the phase plane $\Gamma$ with coordinates $p, q$ and Poisson brackets (PB)

$$\{q, p\} = 1.$$  

(1.2)

If the polar angle $\varphi$ is introduced by

$$p = \sqrt{2h} \sin \varphi, \quad q = \sqrt{2h} \cos \varphi,$$

(1.3)

then for the complex variables $a$ and $a^*$

$$a = \frac{q + ip}{\sqrt{2}}, \quad a^* = \frac{q - ip}{\sqrt{2}}$$

(1.3')

is equivalent to the representation

$$a = \sqrt{h} \exp(i\varphi), \quad a^* = \sqrt{h} \exp(-i\varphi).$$

(1.3'')

1991 Mathematics Subject Classification. 81Q10, 81Q60.

Key words and phrases. Creation and annihilation operators, fermion operators, commutation relations.

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Although the variable $\varphi$ is not global ($\varphi \in S^1$), it is clear that the functions $\exp(\pm i\varphi)$ are correctly defined on the whole phase space $\Gamma$ (except the origin), and for the PB with hamiltonian (1.1) we have

$$\{h, \exp(\pm i\varphi)\} = \pm i\exp(\pm i\varphi) \quad (1.4)$$

which in local coordinates corresponds to

$$\{h, \varphi\} = 1. \quad (1.4')$$

In the quantum case for the operator $h$ (we use the boldface notation) we choose the normal ordering

$$h = a^* a = \frac{p^2 + q^2}{2} - \frac{1}{2} I. \quad (1.5)$$

Then this $h$ can be considered a particle (boson) number operator. It is well known that the eigenvalues of $h$ are nonnegative integers. The corresponding eigenvectors $|n\rangle$ ($n = 0, 1, 2, \ldots$)

$$h|n\rangle = n|n\rangle \quad (1.5')$$

can be constructed by applying the creation operator to the vacuum state $|0\rangle$

$$|n\rangle = \frac{1}{\sqrt{n!}} a^*|^0\rangle$$

and we have

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^*|n\rangle = \sqrt{n+1}|n+1\rangle. \quad (1.5'')$$

The vectors $|n\rangle$ form the basis of the Hilbert space. For further convenience we denote this Hilbert space by $H_B$ (bosonic space).

From the correspondence of PB with commutators, for the phase operators $\exp(\pm i\varphi)$ we get (see (1.4))

$$[h, \exp(\pm i\varphi)] = \mp \exp(\pm i\varphi).$$

So the operator $\exp(i\varphi)$ ($\exp(-i\varphi)$) now decreases (increases) by 1 the eigenvalues of the operator $h$, and thus we can write

$$\exp(i\varphi)|n\rangle = \begin{cases} |n-1\rangle, & n > 0, \\ 0, & n = 0, \end{cases} \quad \exp(-i\varphi)|n\rangle = |n+1\rangle \quad (1.6)$$

from which by virtue of (1.5') we have

$$\exp(i\varphi) = \frac{1}{\sqrt{h+1}}a, \quad \exp(-i\varphi) = a^* \frac{1}{\sqrt{h+1}}. \quad (1.6')$$