ON THE NUMBER OF REPRESENTATIONS OF INTEGERS BY SOME QUADRATIC FORMS IN TEN VARIABLES

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ABSTRACT. A method of finding the so-called Liouville's type formulas for the number of representations of integers by quadratic forms is developed.

In the papers [4,5] four classes of entire modular forms of weight 5 for the congruence subgroup $\Gamma_0(4N)$ are constructed. The Fourier coefficients of these modular forms have a simple arithmetical sense. This allows one to get sometimes the so-called Liouville's type formulas for the number of representations of positive integers by positive quadratic forms in ten variables.

In the present paper we consider positive primitive quadratic forms

\[ f = a_1(x_1^2 + x_2^2) + a_2(x_3^2 + x_4^2) + a_3(x_5^2 + x_6^2) + a_4(x_7^2 + x_8^2) + a_5(x_9^2 + x_{10}^2). \]  

(1)

For the purpose of illustration we obtain a formula for the number of representations of positive integers by the form (1) for $a_1 = \cdots = a_4 = 1$, $a_5 = 4$.

In a similar way one can investigate as well other forms of the kind (1).

As is well known, Liouville obtained in 1865 the corresponding formula for $a_1 = \cdots = a_5 = 1$ only.

1. SOME KNOWN RESULTS

1.1. In this paper $N, a, d, k, n, q, r, s, \lambda$ denote positive integers; $b, u, v$ are odd positive integers; $p$ is a prime number; $\nu, l$ are non-negative integers; $H, c, g, h, j, m, x, y, \alpha, \beta, \gamma, \delta$ are integers; $i$ is an imaginary unit; $z, \tau$ are
complex variables \((\text{Im} \, \tau > 0)\); \(e(z) = \exp 2\pi iz\); \(Q = e(\tau)\); \((\frac{\sqrt{b}}{q})\) is the generalized Jacobi symbol. Further, \(\sum_{h \mod q}\) and \(\sum'_{h \mod q}\) denote respectively sums in which \(h\) runs a complete and a reduced residue system modulo \(q\).

Let

\[
S(h, q) = \sum_{j \mod q} e\left(\frac{hj^2}{q}\right) \quad \text{(Gaussian sum)},
\]

\[
c(h, q) = \sum_{j \mod q} e\left(\frac{hj^2}{q}\right) \quad \text{(Ramanujan's sum),}
\]

\[
\vartheta_{gh}(z|\tau; c, N) = \sum_{m \equiv c \pmod{N}} (-1)^{h(m-c)/N} e\left(\frac{1}{2N} \left(m + \frac{g}{2}\right)^2 \tau\right) e\left((m + \frac{g}{2})z\right)
\]

(1.3)

(\text{theta-function with characteristics } g, h),

hence

\[
\frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) = (\pi i)^n \sum_{m \equiv c \pmod{N}} (-1)^{h(m-c)/N} (2m + g)^n \times
\]

\[
e\left(\frac{1}{2N} \left(m + \frac{g}{2}\right)^2 \tau\right) e\left((m + \frac{g}{2})z\right).
\]

(1.4)

Put

\[
\vartheta_{gh}(\tau; c, N) = \vartheta_{gh}(0|\tau; c, N),
\]

\[
\vartheta_{gh}^{(n)}(\tau; c, N) = \left. \frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) \right|_{z=0}.
\]

(1.5)

It is known (see, for example, [3], p. 112, formulas (2.3) and (2.5)) that

\[
\vartheta_{gh}(\tau; c, N) = \vartheta_{gh}(\tau; c + j, N),
\]

\[
\vartheta_{gh}^{(n)}(\tau; c, N) = \vartheta_{gh}^{(n)}(\tau; c + j, N),
\]

\[
\vartheta_{gh}(\tau; c + N_j, N) = (-1)^{hj} \vartheta_{gh}(\tau; c, N),
\]

\[
\vartheta_{gh}^{(n)}(\tau; c + N_j, N) = (-1)^{hj} \vartheta_{gh}^{(n)}(\tau; c, N).
\]

(1.7)

From (1.3), in particular, according to the notations (1.5), it follows that

\[
\vartheta_{gh}(\tau; 0, N) = \sum_{m=-\infty}^{\infty} (-1)^{hm} Q^{(2Nm+g)^2/8N},
\]

(1.8)

\[
\vartheta_{gh}^{(n)}(\tau; 0, N) = (\pi i)^n \sum_{m=-\infty}^{\infty} (-1)^{hm}(2Nm + g)^n Q^{(2Nm+g)^2/8N}.
\]

(1.9)