Short Communications / Kurze Mitteilungen

A Remark About the Convergence of Interval Sequences

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Abstract — Zusammenfassung

A Remark on the Convergence of Interval Sequences. By some interval iterations the condition $\sigma(r) < 1$ will be assumed for the convergence of the corresponding interval sequence by which $\sigma$ denotes the spectral radius of $r$, and $r$ is a nonnegative Lipschitz matrix. In this paper the theorems are extended for the case $\sigma(r) = 1$.

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Eine Bemerkung zur Konvergenz von Intervallfolgen. Bei manchen Intervalliterationen wird zur Konvergenz einer entsprechenden Intervallfolge vorausgesetzt, daß $\sigma(r) < 1$ ist, wobei $\sigma$ den Spektralradius von $r$ bezeichnet und $r$ eine nichtnegative Lipschitzmatrix ist. In dieser Arbeit werden die Aussagen auf den Fall $\sigma(r) = 1$ erweitert.

1. Introduction

There are many papers dealing with interval sequences, which are defined with the help of interval operators. With regard to references we refer to the bibliography [1]. For the point-convergence of such an interval sequence to a solution $x^*$ of an equation it will generally be assumed that a Lipschitz constant or (in case of a pseudometric Lipschitz condition) the spectral radius of a Lipschitz matrix is less than 1.

In the following paper it will be shown, that by applying the iteration method: $X_{k+1} := F(X_k)$ for one class of interval operators the usual existence test $F(X) \subseteq X$ is sufficient for the convergence of the sequence of midpoints $\bar{x}_k$ to a solution $x^*$ of the given equation.

For the class of interval-NEWTON-operators even the interval sequence $\{X_k\}$ itself converges to $x^*$.

If we apply the iteration method: $X_{k+1} := F(X_k) \cap X_k$, if additionally the spectral radius of the corresponding Lipschitz matrix is equal to 1 and if the Lipschitz matrix
is irreducible, then an analogous statement is true. Corresponding to the choice of the interval operator it holds \( \lim_{k \to \infty} \bar{x}_k = x^* \) resp. even \( \lim_{k \to \infty} X_k = x^* \).

With regard to notations we refer to section 2 of [2].

### 2. Problem

Let \( g \mid X_0 \in \mathbb{I} (\mathbb{R}^n) \to \mathbb{R}^n \) be a function, satisfying a Lipschitz condition

\[
g(x_1) - g(x_2) \leq L(x_1 - x_2) \quad \text{for all } x_1, x_2 \in X_0
\]

with \( L = [l, L] \in \mathbb{I} (\mathbb{R}^{n \times n}) \).

Under the assumption that

\[
T := \frac{1}{2} (J + \overline{I}) \in \mathbb{R}^{n \times n}
\]

is regular, it exists

\[
a := T^{-1}
\]

and

\[
r := \frac{1}{2} |a| |(I - J)|.
\]

\( r \) is a Lipschitz matrix of a pseudometric Lipschitz condition of the NEWTON-transformation

\[
f(x) := x - ag(x),
\]

because

\[
|f(x_1) - f(x_2)| \leq r |x_1 - x_2| \quad \text{for all } x_1, x_2 \in X_0
\]

holds.

It is known that \( x^* \in X_0 \) on account of (3) is a fixpoint if and only if \( x^* \) is a solution of the equation \( g(x) = 0 \).

For the inclusion of such a solution we can define interval sequences \( \{X_k\} \) with the help of several interval operators. At that we distinguish two classes of operators with different properties: \( K \)-operators and interval-NEWTON-operators. For the following we want to investigate the convergence behaviour of the interval sequences defined by the different operators for each one representative of these two classes.

### 3. The \( K \)-Operator

Let the interval function \( K \mid \mathbb{I} (X_0) \to \mathbb{I} (\mathbb{R}^n) \) with

\[
K(x) := \bar{x} - ag(\bar{x}) + [-r, r](X - \bar{x}), \quad \bar{x} := \frac{1}{2} (x + \bar{x})
\]

be continuous and inclusion isotone.

For the interval sequence \( \{X_k\} \) defined by

\[
X_{k+1} := K(X_k), \quad k = 0, 1, \ldots
\]

holds the following