SMOOTH CONVEX LIMIT SURFACES IN THE SPACE OF
SYMMETRIC SECOND-RANK TENSORS

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Equations are proposed to describe smooth (regular) convex hexagonal and triangular curves in the deviatoric plane. The equations are used to construct smooth convex limit surfaces for media whose properties are arbitrarily anisotropic. The results supplement the findings in [A. Lagzdin' and A. Zilauts, Mekh. Kompozitn. Mater., 32, No. 3, 339-349 (1996)], where the third invariant of a tensor was introduced into the limit-surface equation by means of singular deviatoric curves.

A method was proposed in [1] for constructing limit surfaces in the space of symmetric second-rank tensors. The method involves the combination — nonlinear and linear — transformation of a unit sphere and an elementary square paraboloid. The sufficient conditions for convexity of the limit surfaces were also given in that study. However, due to certain features of the introduction of the third invariant of a tensor into the equations of surfaces, singularities are formed on the surfaces in the form of points of inflection. A normal to the given surface suddenly changes direction when it passes through such points. In a number of cases — such as the theory of strength, for example — the singularities pose no problem. However, there are applications, including the theory of plasticity, in which their presence is undesirable. Thus, we need to know how to remove them if necessary. The present article is devoted to the resolution of this problem.

We will examine a 6D space of dimensionless symmetric second-rank tensors $p = p_{i j} e_i e_j$, $i, j = 1, 2, 3$. We designate the deviatoric part of the tensors $p$ as $\text{Dev } p$, and we designate their invariant parts as $I_i(p)$ and $I_{iD}(p) = I_i(\text{Dev } p)$, respectively:

$$I_i(p) = p_i = \sqrt{3} p_0; \quad I_{iD}(p) = p_i p_0; \quad I_j(p) = p_j p_0.$$

As in [1], we construct three rotational surfaces in $p$-space. The rotational axis of these surfaces coincides with the hydrostatic axis $p_0$ of this space:

$$v_1(p) = I_{1D}(p) - \lambda_1^2 (1 - p_0^2) = 0; \quad v_2(p) = I_{2D}(p) + k \lambda_2^2 p_0 = 0; \quad v_3(p) = I_{3D}(p) + k p_0^2 = 0,$$

where $k = \pm 1$, $\lambda_1 = \lambda_1(p_0)$.

Two variants are proposed for each function $\lambda_i$:

$$\lambda_1 = \lambda_1(p_0) = \frac{1 + \lambda (1 - k p_0)}{1 + \lambda^2 (1 - k p_0)^2}; \quad \lambda_1 = \lambda_1(p_0) = \frac{1 + l (m + \lambda_0 (1 - k p_0))}{(1 + \lambda^2 (1 - k p_0)^2)^{1/2}}, \quad m \geq 1/2,$$

where $l, n \geq 0; k = \pm 1$.
The equation \( v_1 = 0 \) describes a closed regular (smooth) conoidal surface with its center at the origin of the coordinates. As can easily be seen, the surface is obtained from the nonlinear transformation of a unit sphere \( I_2(p) = 1 \). The transformation is realized by dividing all of the deviatoric components of the tensor \( p \) by the scalar function \( \lambda_1(p_0) \).

In the case \( \lambda_1 = \lambda_{11} \), the surface \( v_1 = 0 \) is convex if

\[
0 \leq m \leq 1/2
\]

or

\[
-1/4 \leq m < 0; \quad \frac{6m}{4m^2 - 1} \leq \tau \leq \frac{1}{2m + 1},
\]

while in the case \( \lambda_1 = \lambda_{12} \) it is convex if

\[
n = 0
\]

or

\[
n = \frac{8l(lm + 1)}{3(2l + 1)}.
\]

The equation \( v_2 = 0 \) defines a regular (smooth) parabolic surface with its vertex at the point \( p_0 = 0 \). It is obtained by nonlinear transformation of an elementary square paraboloid.

In the case \( \lambda_2 = \lambda_{21} \), the surface \( v_2 = 0 \) is convex if condition (1) is satisfied. In the case \( \lambda_2 = \lambda_{22} \), it is convex if

\[
0 \leq n \leq 4/3l(lm + 1).
\]

The surface \( v_3 = 0 \) is a paraboloid of the order \( 2m \). The surface is always convex, and if \( m > 1/2 \) it is also regular. When \( m = 1/2 \), instead of a paraboloid we have a cone with the point of inflection \( p_0 = 0 \).

We introduce the third deviatoric invariant \( I_{3D}(p) \) into the equation \( v_1 = 0 \). We use \( p_1, p_{II}, p_{III} \) as the principal values of the tensor \( p \) satisfying the typical condition that \( p_1 \geq p_{II} \geq p_{III} \). In the p-space we make the change of variables to \( z_1, z_2, z_3 \) using

\[
\frac{p_1 - p_{II}}{\sqrt{2}} = z_1; \quad \frac{p_{II} - p_{III}}{\sqrt{2}} = \frac{1}{2}(z_1 - \sqrt{3}) z_2; \quad \frac{p_{III} - p_1}{\sqrt{2}} = \frac{1}{2}(z_1 + \sqrt{3}) z_2; \quad p_0 = z_3,
\]

and we examine the section of the surface \( v_1 = 0 \) with the deviatoric plane \( z_3 = \text{const} \). In this plane, the surface leaves a trace in the form of a circle called a von Mises circle:

\[
I_{3D}(p) = z_1^2 + z_2^2 = c^2.
\]

Let \( \kappa \) be a certain scalar function of \( I_{3D}(p) \) or some parameter dependent on \( I_{3D}(p) \). We transform circle (6) by multiplying the deviatoric component of \( p \) by \( \kappa \). Then instead of (6) we obtain a new equation:

\[
\kappa I_{3D}(p) = c^2.
\]

If \( \kappa \neq \text{const} \), then the \( z_3 \) axis for figure (7) may be a rotation axis only of the sixth or third order. For the sake of brevity, we will call deviatoric curve (7) a hexagonal curve in the first case and a triangular curve in the second case.

As the argument for the function \( \kappa \), we use the Lode—Nadai parameter \( \mu(p) \) or the deviatoric angle \( \omega(p) \):

\[
\mu(p) = \frac{(p_1 - p_{III}) - (p_1 - p_{II})}{p_{III} - p_1} = \sqrt{3} \; \tan \omega;
\]

\[
\omega(p) = 1/3 \; \arcsin \left[ -\sqrt{3} \right] \frac{I_{3D}(p)}{I_{3D}^2(p)};
\]

\[-1 \leq \mu \leq 1; \quad -\pi/6 \leq \omega \leq \pi/6.\]