SPECTRUM OF THREE-DIMENSIONAL LANDAU OPERATOR PERTURBED BY A PERIODIC POINT POTENTIAL

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A study is made of a three-dimensional Schrödinger operator with magnetic field and perturbed by a periodic sum of zero-range potentials. In the case of a rational flux, the explicit form of the decomposition of the resolvent of this operator with respect to the spectrum of irreducible representations of the group of magnetic translations is found. In the case of integer flux, the explicit form of the dispersion laws is found, the spectrum is described, and a qualitative investigation of it is made (in particular, it is established that not more than one gap exists).

Introduction. In this paper, we consider the Hamiltonian of a charged particle (charge $e$, mass $m^*$) in a magnetic field $B$, i.e., the Schrödinger operator with magnetic field

$$H_0 = \frac{1}{2m^*}(\hat{p} - eB)^2$$

in the space $L^2(\mathbb{R}^3)$ [for the vector potential $A$, we choose the symmetric gauge $A = (B \times r)/2$; in the physics literature, $H_0$ is also called the Landau operator]. We shall be interested in the spectrum of periodic perturbations $H$ of this operator: $H = H_0 + V$. Despite the steadily growing interest of mathematicians in the spectral theory of periodic Landau operators (see, for example, [1--3]), several basic questions of this theory are still unresolved even in the case of a rational flux of the field through the boundary of the unit cell of the lattice of periods $A$ of the potential $V$. For example, for the two-dimensional periodic Landau operator with nontrivial smooth potential, the conjecture that there are no eigenvalues in its spectrum is still unproved [4].

In the case of a rational flux, a magnetic analog of the Bloch analysis of the periodic Landau operator is possible [1,2]. It leads to a band picture of the spectrum $\sigma(H)$ of the operator $H$ (we note that by an enlargement of the lattice $A$ this case can be reduced to the consideration of integer flux through the boundary of the unit cell). For such a structure of the spectrum, the question of the number of gaps in the spectrum $\sigma(H)$ is important. If $A = 0$, then the well-known Bethe–Sommerfeld hypothesis asserts that the number of gaps in $\sigma(H)$ is finite [5]; recently, this hypothesis was established by Skriganov [6,7]. His results can be generalized to the case of Schrödinger and Dirac operators with periodic vector potential $A$ [8]. We note immediately that, in contrast to the case $A = 0$, for two-dimensional Landau operators the Bethe–Sommerfeld hypothesis is trivially false. Indeed, if $|V(x, y)| \leq h\omega/3$, where $\omega = |eB|/cm^*$ is the cyclotron frequency, then the Landau levels spread into nonintersecting bands, and therefore their number is infinite. For another, less trivial example of this kind, see [9].

One of the results of the present paper confirms the Bethe–Sommerfeld hypothesis for perturbation of the three-dimensional Landau operator by a periodic point potential [10,11] (we note that for Schrödinger operators ($A = 0$) with point perturbation the Bethe–Sommerfeld hypothesis was proved in [12,13]). This result is based on explicit expressions obtained in the paper for the Green's function of the operator $H$ and explicit expressions for its dispersion laws, which follow from the energy and spatial asymptotic behaviors of the Green's function of the Hamiltonian $H_0$. The main results of the paper are based on the technique of decomposition of the state space with respect to the spectrum of irreducible representations of the group of magnetic translations (this group was introduced in [14]); the two-dimensional form of this technique was used to investigate the spectrum of the periodic Landau operator in [15--17]. The decomposition that we use is an analog of the decomposition of $L^2(\mathbb{R}^3)$ into a direct integral over the Brillouin zone in the $p$ representation [18].

1. We here introduce the basic notation and briefly describe the construction of the decomposition of the state space $L^2(\mathbb{R}^3)$ into a direct integral with respect to the spectrum of irreducible representations of the group of magnetic translations, i.e., over the magnetic Brillouin zone. It should be noted that at the physical level of rigor the use of direct integrals reduces to transition to the representation of Wannier quantum numbers [19]; this is something that, essentially, was already done by Zak [20].

Let $\Lambda$ be a lattice in $\mathbb{R}^3$ with generators $a_1, a_2, a_3$. In what follows, we shall denote the coordinates of a vector $v$ with respect to the basis $a_1, a_2, a_3$ by numerical indices: $v = v_1a_1 + v_2a_2 + v_3a_3$, and with respect to the standard basis $i, j, k$ by letter indices: $v = v_i + v_j + v_k$. To go over to dimensionless units, we introduce the vectors $\xi = eB/2\pi c\hbar$ and $\eta = \Omega\xi$, where


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\( \Omega = a_1(a_2 \times a_3) \) is the oriented volume of the unit cell \( C_A \) of the lattice \( \Lambda \).

\[
C_A = \{ t_1a_1 + t_2a_2 + t_3a_3 : 0 \leq t_1, t_2, t_3 < 1 \}.
\]

The coordinates of the vector \( \eta \) in the basis \( a_1, a_2, a_3 \) are the number of flux quanta of the field \( B \) through the corresponding boundary of the cell \( C_A \): \( \eta = \xi(a_2 \times a_3) \), ... (the quantum of magnetic flux is \( \Phi_0 = |e/2\pi c| \)). The field \( B \) is said to be rational (with respect to the lattice \( \Lambda \)) if all the numbers \( \eta \) are rational \([14,21]\); obviously, this definition does not depend on the choice of the lattice generators. Moreover, if the condition of rationality is satisfied the generators \( a_1, a_2, a_3 \) can be chosen in such a way that \( \eta_1 = \eta_2 = 0 \); in what follows, we assume that the choice of the generators is made in this way and we denote \( \eta_3 = \eta, \quad \xi_3 = \xi \). In addition, we choose \( i, j, k \) in such a way that \( a_1y = a_3x = a_2 = 0, \quad a_{1x}, a_{3y} > 0 \).

Using the notation we have introduced, we can express the Hamiltonian \( H_0 \) up to a factor \( \hbar^2/2m^* \), which in what follows we omit, in the form

\[
H_0 = \left( -i \partial/\partial x + \pi \xi y \right)^2 + \left( -i \partial/\partial y - \pi \xi x \right)^2 - \partial^2/\partial z^2.
\]

By \( W(\xi) \) we denote the group of magnetic translations, i.e., the set \( R^3 \times S \), where \( S = \{ x \in C: \| x \| = 1 \} \), equipped with the multiplication

\[
(v, \zeta)(v', \zeta') = (v + v', \zeta \zeta' \exp \left[ \pi i \xi (v \times v') \right])
\]

(see \([14]\)). Let \( (v, \zeta) \in W(\xi) \); by \( [v, \zeta] \) we denote the operator that acts on functions \( f \) defined in \( R^3 \) in accordance with

\[
[v, \zeta] f(r) = \zeta \exp \left[ -\pi i \xi (r \times v) \right] f(r - v).
\]

It is readily verified that the mapping \( (v, \zeta) \mapsto [v, \zeta] \) is a unitary representation of the group \( W(\xi) \) in the space \( L^2(R^3) \) with respect to which the operator \( H_0 \) is invariant.

Let \( V \) be a periodic perturbation of \( H_0 \) with lattice of periods \( \Lambda \); then \( H = H_0 + V \) remains invariant with respect to the operators \([\lambda, \xi]\), where \( \lambda \in \Lambda \). We denote by \( W(\xi, \Lambda) \) the subgroup of the group \( W(\xi) \) consisting of all elements of the form \( (\lambda, \xi) \), where \( \lambda \in \Lambda \) and

\[
\zeta = \exp(\pi i \sum n_k \eta_k), \quad n_k \in Z.
\]

If one of the numbers \( \eta_j \) is irrational, then the unitary representations of the group \( W(\xi, \Lambda) \) can be decomposed in a nonunique manner into irreducible representations \([22]\) [i.e., \( W(\xi, \Lambda) \) is a group of type II], and harmonic analysis on this group does not bring a significant gain in the investigation of the operator \( H \). In this connection, the assumption of rationality of the magnetic field with respect to the lattice \( \Lambda \) is a standard assumption of the theory of periodic Landau operators \([1,14,21]\), and we shall also adopt it in what follows. Henceforth, we shall assume that the rational number \( \eta \) is positive and given by an irreducible fraction: \( \eta = N/M \) \((N, M \in N)\).

With the basis \( a_1, a_2, a_3 \) there is associated a complete orthonormal set of (generalized) eigenfunctions \( \psi_0(r; l, q, s) \) (Landau functions). This set depends on three parameters \( l, q, s \) \((l \in N, q, s \in R)\), and the form of the functions is \([22]\)

\[
\psi_0(r; l, q, s) = A(l, q, s) \exp \left[ 2\pi i \left( a_2 z + \xi x (y/2 + \beta q + \gamma s) \right) \right] u_l \left[ (2\pi |\xi|)^{1/2} (y + \beta q + \gamma s) \right],
\]

where \( u_l(x) = \exp(-x^2/2)H_l(x) \) is an Hermite function;

\[
\beta = a_2 y / \eta; \quad \gamma = -a_1 z a_2 y / a_3 z \eta; \quad A(l, q, s) = (2|\xi|)^{1/4} \exp \left[ \pi i (\sigma_1 q^2 + \sigma_2 q s) \right] (a_1 z a_2 z l!)^{-1/2};
\]

\[
\sigma_1 = a_2 z / a_1 x \eta; \quad \sigma_2 = 2 (a_1 z a_2 z - a_1 x a_2 x) / a_1 x a_2 z \eta.
\]

We denote \( \alpha = 4\pi^2 / a_2 z \); then

\[
H_0 \psi_0(r; l, q, s) = (\epsilon_l + \alpha s^2) \psi_0(r; l, q, s),
\]

where \( \epsilon_l = 4\pi |\xi| (l + 1/2) \) are the Landau levels.

We introduce the Fourier—Landau transformation \( F_L \), which establishes a Hilbert isomorphism between the state space \( L^2(R^3) \) and the space \( \hat{P}(N) \otimes L^2(R^2) \) in accordance with

\[
(F_L f)(l, q, s) = \int_{R^3} f(x) \psi_0^*(x; l, q, s) dx,
\]

and for the action of the operator \([v, \zeta]\) we then obtain \([22]\)