Finite Presentation of Alternating Groups

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Abstract. Graham Higman posed the question: How small can the integers $p$ and $q$ be made, while maintaining the property that all but finitely many alternating and symmetric groups are factor groups of $\Delta(2, p, q) = \langle x, y : x^2 = y^p = (xy)^q = 1 \rangle$? He proved that for a sufficiently large $n$, the alternating group is a homomorphic image of the triangle group $\Delta(2, p, q)$ where $p = 3$ and $q = 7$. Later, his result was generalized by proving the result for $p = 3$ and $q \geq 7$. Choosing $p = 4$ and $q \geq 17$ in this paper we have answered the "Higman Question".

§1. Introduction

The group $[p, q]$ or $\{q, p\}$, defined by the relations $x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^2 = 1$, is the group whose fundamental region is a triangle with angles $\pi/p$, $\pi/q$ and $\pi/2$. This group is the complete symmetry group of either of the two reciprocal polyhedra $\{p, q\}$, $\{q, p\}$. Its subgroup $[p, q]^+$ is isomorphic to the triangle group $\Delta(2, p, q) = \langle x, y : x^2 = y^p = (xy)^q = 1 \rangle$. Of course, $[p, q]^+$ is of index 2 in $[p, q]$, and is finite if $p < 3$ and $q \leq 6$. The case for $p = 3$ and $q = 7$, which is studied extensively in [3], [6] and [8], becomes a special case, because these are the smallest values for which the group $L_\Delta(2, 3, 7)$, known as the Hurwitz group, becomes an infinite insoluble group.

The groups $G^{p,q,m} = \langle x, y, t : x^2 = y^p = t^2 = (xt)^2 = (yt)^2 = (xy)^q = (xyt)^m = 1 \rangle$ are symmetric groups of regular maps $\{p, q\}_m$, [4]. Let $p$ and $q$ be two points, at a distance $m$ apart along a Petrie path. Then the map $\{p, q\}_m$ is constructed from the tessellation $\{p, q\}$ of the hyperbolic plane by identifying these points. The extended triangle group $\Delta^*(2, p, q)$, with presentation $\langle x, y, t : x^2 = y^p = t^2 = (xt)^2 = (yt)^2 = (xy)^q = 1 \rangle$, has $G^{p,q,m}$ as its factor group. The coset diagrams, defined for $G^{p,q,m}$, are effectively used by Graham Higman in his result (which he never published but reference to which is available in [5] and [7]) that for all but finitely many positive integers $n$, the alternating group $A_n$ is a Hurwitz group. He proved that for a sufficiently large $n$, the group $A_n$ is a homomorphic image of the triangle group $\Delta(2, p, q)$ where $p = 3$ and $q = 7$. Later, his result was generalized by M. Conder [1], [2] who proved the same result for $p = 3$ and $q \geq 7$. Choosing $p = 4$ and $q \geq 17$ we have answered the 'Higman Question' as follows: for all $n > 34 \times 35$, where $n = 34i + 35j$ with $i, j > 0$, $A_n$ is a homomorphic image of $\Delta(2, 4, q)$ with presentation $\langle x, y : x^2 = y^p = (xy)^q = 1 \rangle$, $q \geq 17$.

Not much is known about the groups $\Delta^*(2, 4, q)$ and $\Delta(2, 4, q)$. For some values of $q$ the authors are aware of only the following information.

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The group $\Delta(2, 4, 17)$, which is isomorphic to $[4, 17]^{+}$, is an infinite group. The groups $\Delta(2, 4, 2) \cong D_5$ and $\Delta(2, 4, 3) \cong S_4$ are finite groups. The triangle group $\Delta(2, 4, 4)$ is Abelian-by-cyclic. Apart from this, not much is known about the groups $\Delta(2, 4, q)$ for $q \geq 5$.

§2. Actions of $\Delta^*(2, p, q)$ on Finite Sets

A coset diagram is a graphical representation of a permutation action of a finitely generated group. By $D(n)$, we mean the coset diagram depicting the action of the group $G(2, p)$, satisfying the relations

$$x^a = y^p = t^2 = (xt)^2 = (yt)^2 = 1,$$

on the set of $n$ vertices. We choose the generators $x, y$ and $t$, for the group $G(2, p)$. The permutations induced by these generators will be represented as follows.

The $p$ cycles of $y$ are denoted by $p$-gons whose vertices are permuted anti-clockwise by $y$. Similarly fixed points of $x$ and $y$ are denoted by heavy dots. Any two vertices interchanged by the involution $x$ are joined by an edge (not necessarily a straight line). The action of $t$ is given by reflection in a vertical axis of symmetry.

In a coset diagram, depicting a permutation representation of $\Delta^*(2, p, q)$, two vertices $a, b$ are said to form a $k$-handle if and only if $a, b$ are both fixed by $x$, and $(xy)^k$ and $t$ both map $a$ to $b$. We denote them by $(a, b)_k$ as defined in [8]. Suppose $D(m)$ and $D(n)$ are two coset diagrams for $\Delta^*(2, p, q)$ containing the $k$-handles $(a, b)_k$ and $(a', b')_k$ respectively. We use the method given in [1] to join $D(m)$ to $D(n)$ to get a new diagram $D(m+n)$. In order to do so we add two new $x$-edges, joining $a$ to $a'$ and $b$ to $b'$. The new diagram $D(m+n)$ is also the coset diagram depicting the homomorphic image of the group $\Delta^*(2, p, q)$. For, we can easily check that the relations (*) are still satisfied. Also if $(ac_1c_2c_3c_4bc_5c_6\cdots c_{q-2})$ and $(ad_1d_2d_3d_4d_5\cdots d_{q-2})$ are cycles of $xy$ in the representation of $\Delta^*(2, p, q)$ depicted by $D(m)$ and $D(n)$ respectively, then in $D(m+n)$ we observe that $(ad_1d_2d_3d_4b'c_5c_6\cdots c_{q-2})$ and $(a'c_1c_2c_3c_4bd_5d_6\cdots d_{q-2})$ are cycles of $xy$. The rest of the cycles of $xy$ will remain unchanged and so every vertex of $D(m+n)$ is fixed by $(xy)^q$. Thus $D(m+n)$ depicts an action of $G(2, p)$ on $m+n$ vertices.

§3. Homomorphic Images of $\Delta(2, 4, q), q \geq 17$

To prove the main result we need the following theorem from [9].

**Theorem 3.1** (C. Jordan, 1873). Let $p$ be a prime and $G$ a primitive group of degree $n = p+k$ with $k \geq 3$. If $G$ contains an element of order and degree $p$, then $G$ is either alternating or symmetric.

Here we are concerned with the actions of the group $G(2, 4) = \langle x, y, t : x^2 = y^4 = t^2 = (xt)^2 = (yt)^2 = 1 \rangle$ on $n$ points to prove the following result.

**Theorem 3.2.** For all $n > 1190$, where $n = 34i + 35j$, with $i, j > 0, A_n$ is a homomorphic image of $\Delta(2, 4, 17) = \langle x, y : x^2 = y^4 = (xy)^{17} = 1 \rangle$.

**Proof.** We consider the actions of the group $G(2, 4)$ on 34 and 35 points. Let $D(34)$ and $D(35)$ be the coset diagrams for these actions.

Consider the two coset diagrams: